A nonsingular Model of the Universe

Mohammad Saleh Abdul-latif

2008

بسم ا الر ن الرحيم

قُلْ هلْ يستوِي الذِين يعلَمونَ والذِين لَا يعلَمونَ إِنما يتذَكر أُولُوا الْأَلْبابِ

صدق ا العظيم

قال رسول ا ﷺ: من سلك طَريقاً يلْتمِس فِيهِ عِلْماً سهلَ ا ُ له بِهِ طَرِيقاً إ ا نةِ

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September 2008

Discussion Committee:

Dr. Yacoub Anini (Supervisor)
Dr. Henry Jaqaman (Member)
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Name	Signature
Dr. Yacoub Anini (Supervisor)	…………..
Dr. Henry Jaqaman (Member)	Henry Jaqaman …………..
Mr. Ramzi Rihan (Member)	…………..

إهداء

إلى هذا الوطن الجريح الذي طال ليله.

إلى الشهداء، قناديل في طريق الحرية.

إلى زوجتي, نور حياتي.

إلى سنا ظلمتي والتي أقبلت على الدنيا في المراحل الأخيرة من إنجازهذا العمل لتضئ لي طريق النهاية....إلى ابنتي ''سنا'' نور حياتي.

إلى أبي...أمي...أخواني و أخواتي وكل عائلتي.

أهدي هذا العمل المتواضع

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ملخص

هناك بعض النظريات الفيزيائية تتنبأ بوجود نقاط في الكون متناهية في الصغر(singularities) بحيث يكون حجمها مساويا للصفر تقريبا ويكون إنحناؤها(تقوسها) كبيرا جدا. تشكل هذه النقاط مشكلة في علم الفيزياء. ومن النظريات التي تتنبأ بوجود مثل هذه النقاط النظرية النسبية العامة حيث أن هذه النظرية لا تقدم وصفا كافيا لسلوك الكون في مناطق التقوس الكبير. علاوة على ذلك, هناك مشاكل عديدة في علم الكون كمشكلة الأفق ومشكلة إنكماش النجوم العظيمة عندما تتقلص هذه النجوم لتكون ثقوبا سوداء متناهية في الصغر و كذلك مشكلة الإنفجار العظيم التي تنص على أن الكون بدأ من نقطة متناهية في الصغر أخذت في التمدد.

كل هذه الإعتبارات تدفعنا الى تقديم فرضية جديدة تقدم حلولا لهذه المشاكل و تقدم نموذجا أكثر واقعية لسلوك الكون كما تقدم حدا للإنحناء بحيث أن إنحناء أي نقطة في الكون لا يتجاوز هذا الحد, هذه الفرضية هي" فرضية الإنحناء المحدود".

لتنفيذ هذه الفرضية قمنا بعمل تعديل على النظرية النسبية عن طريق إدخال قيمة قصوى للإنحناء في معادلة آينشتاين التالية:

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = -8\pi G T^{\mu\nu}$$

حيث أصبحت هذه المعادلة بعد إدخال القيمة القصوى للتقوس Λ كما يلي:

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} - \frac{1}{4}\Lambda\left(1 - \sqrt{1 - \frac{R^2}{\Lambda^2}}\right) g^{\mu\nu} = -8\pi G T^{\mu\nu}$$

كما نلاحظ من المعادلة فإنه في المناطق قليلة التقوس تكون R صغيرة جدا وهذا يؤدي إلى إختزال المعادلة لتؤول الى معادلة آينشتاين.

في الفصل الثاني قدمنا المعادلة المعدلة ووجدنا من مركباتها الزمانية والمكانية معادلات تفاضلية من الدرجة الأولى و الثانية تصف سلوك الكون عندما سيطر عليه الإشعاع (radiation dominated universe) و

سلوكه عندما تسيطر عليه المادة(matter dominated universe) و كذلك تصف سلوك الكون حسب الشكل الهندسي له, كذلك وجدنا كيف يتغير حجم نجمة عظيمة مع الزمن عندما تنكمش هذه النجمة لتشكل ثقبا أسود. وفي الفصل الثالث وجدنا حلولا باستخدام الحاسوب للمعادلات التفاضلية ثم قمنا برسم هذه الحلول.

Abstract

The infinities or singularities (points of spacetime where the curvature blows up) are considered as serious problems in physics. Classical general relativity predicts spacetime singularities. This theory does not give an enough description of the behavior of the spacetime in the high curvature regions. On the other hand, in quantum field theory there are other kinds of singularities coming from the non-renormalizability of this theory. Moreover, there are several problems in cosmology such as the horizon and flatness problem.

All these considerations point in the direction of the Limiting Curvature Hypothesis (LCH). This hypothesis provides natural solutions to gravitational singularities and introduces a more realistic cosmological model. According to this hypothesis, the curvature of spacetime at any point can never be larger that certain limiting value.

In order to implement this hypothesis, we must modify the general relativity by introducing a limiting value for the curvature; this can be done by modifying Einstein’s field equations:

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = -8\pi G T^{\mu\nu}$$

By inserting a cosmological constant Λ, which is the limiting value of curvature, the modified field equations are:

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} - \frac{1}{4}\Lambda(1 - \sqrt{1 - \frac{R^2}{\Lambda^2}})g^{\mu\nu} = -8\pi G T^{\mu\nu}$$

At low curvatures, when R is very small, the new equations are reduced to Einstein's field equations.

In chapter 2, we introduced the modified field equations which satisfy LCH and found first and second order differential equations from the time-time and space-space components of the field equations for both matter and radiation universes and for different kinds of geometries of spacetime, we also obtained nonsingular spherically symmetric solution that represent a giant star when it collapses to form a black hole. In chapter 3, we solved the differential equations numerically and found nonsingular solutions and we plotted these solutions.

Chapter 1

Introduction

It is well known that the most important problems in physics are infinities or singularities. Hawking and Ellis [1] define a singularity as a point where the metric tensor (to be defined later) is undefined or is not suitably differentiable. However the trouble with this is that one could simply cut out such points and say that the remaining space represented the whole of spacetime, which would then be nonsingular according to this definition. Indeed, it would seem inappropriate to regard such singularities as being part of spacetime, because the equations of physics would not hold at them and it would be impossible to make any measurement. Another definition is that singularities are points of the spacetime where the curvature blows up. They are tremendously dense points with approximately zero volume. Classical general relativity (CGR) predicts spacetime singularities as demonstrated by Hawking, Penrose, and Ellis in a set of singularity theorems [1,2]. Hawking and Penrose state that, for reasonable matter content which is free from exotic matter (*exotic matter* is a concept in particle physics which covers any material that violates one or more classical conditions or is not made of known baryonic particles. Such materials would possess qualities like negative mass or negative energy or being repelled rather than attracted by gravity), spacetimes in general relativity are almost guaranteed to be geodesically incomplete (have singularities). The singularity theorems of Hawking and Penrose imply that general relativity is an incomplete description of the behavior of spacetime at high curvatures. As examples, the two most useful spacetimes in general relativity, the Schwarzschild solution describing black holes and the Friedmann-Robertson-Walker (FRW) solution describing homogeneous, isotropic cosmologies, both contain important singularities. On the other hand, in quantum field theory in general and in quantum gravity in particular one is faced with other kinds of singularities coming from the nonrenormalizability of these theories [3]. The big bang and the big crunch in addition to the singularities resulting in the gravitational collapse of massive stars which collapse to form black holes are physical examples of gravitational singularities.

Moreover in cosmology there are several problems, (the horizon, the flatness, the isotropy of microwave background radiation, and the seeds from which galaxies were formed). All these problems found their natural solution by introducing a long period of inflation, during which the universe expanded exponentially and the spacetime has almost de Sitter geometry [4].

All the above considerations point in the direction of the limiting curvature hypothesis [5,6,7,8]. In this hypothesis, CGR will be modified in order to prevent the occurrence of infinities. This hypothesis provides natural solutions to gravitational singularities by introducing a limiting value of the curvature. As a result this hypothesis introduces a reasonable and a realistic cosmological model of the universe since in the real universe there is no meaning to say that something is a singularity and has zero volume. In order to implement this hypothesis we will modify Einstein's gravitational action and field equations then we will find cosmological equations from the new field equations that describe the behavior of the universe for different kinds of matter.

Chapter 2

Theoretical Background

In this chapter, we will talk about classical general relativity and Einstein's field equations, and then we will introduce the limiting curvature hypothesis by modifying Einstein's field equations. After that we will find differential equations from the new field equations that describe the behavior of the universe for different kinds of matter. We will begin by listing some astrophysical terminologies that appear frequently in cosmology and astronomy because of their importance.

2.1 Astrophysical Terminologies

2.1.1 The spacetime

It is a mathematical model that combines space and time into a single model. Spacetime is usually interpreted with space being three-dimensional and time as the fourth dimension.

2.1.2 The big bang

In physical cosmology, the Big Bang is the scientific theory that the universe emerged from a singularity about 13.7 billion years ago. Physicists do not widely agree on what happened before this, although CGR predicts a gravitational singularity. Now what are the major evidences which support the big bang theory? The first evidence is that galaxies appear to be moving away from us at speeds proportional to their distances, this is called "Hubble's Law" which is:

$$v = H_0 r \tag{2.1}$$

where v is the recessional velocity of the galaxy or other distant object, r is the distance to the galaxy or object, and H_0 is Hubble's constant which is equal to 71 ± 7 km/s.Mpc [1] (the value 71 km/s.Mpc is equal to 2.3×10^{-18} s^{-1}). This observation supports the expansion of the universe and suggests that the universe was once compact.

The second piece of evidence is that if the universe was initially very hot, as the big bang theory suggests, we should be able to find some remnant of this heat. In 1965, Penzias and Wilson discovered a 2.7 K the Cosmic Microwave Background radiation (CMB) which fills the entire observable universe. It was generated in the early universe about 300 000 years after the big bang and fills all space almost uniformly. This radiation has the same distribution in wavelength as does radiation in an enclosure whose walls are held at a temperature of 2.7 K. This enclosure

---

[1] The *parsec* (pc) is a unit of length used in astronomy, and its length is based on the method of trigonometric parallax, one of the oldest methods for measuring the distances to stars. It is defined to be the distance from the Earth to a star that has a parallax of 1 arcsecond when the viewing position changes by 1 AU. The actual length of a parsec is approximately 3.086×10^{13} kilometers, 3.262 light-years or 1.918×10^{13} miles)

is the entire universe. The cosmic microwave background radiation is considered as evidence which supports the big bang theory. This is thought to be the remnant which scientists were looking for.

Finally, the abundance of the light elements (Hydrogen and Helium) found in the observable universe is thought to support the big bang theory.

2.1.3 The big crunch

This is the hypothesis that the universe will collapse upon itself after its expansion eventually stops. It is a counterpart to the Big Bang. It may happen if the gravitational attraction of all the matter in the universe were high enough; the expansion of the universe would slow down and then reverse. The universe would then contract and all matter and energy would be compressed into a gravitational singularity.

2.1.4 The horizon problem

Collins [9] defines the horizon problem as a problem that arises from the similarity of conditions in different parts. The microwave background radiation from opposite directions in the sky is characterized by the same temperature which is 2.7 K. Such similarity could only be established by mutual interactions which could never have taken place, because the regions of space from which they were emitted at 500,000 years were more than light transit time apart and could not have "communicated" with each other to establish the apparent thermal equilibrium; they were beyond each other's "horizon". This problem is called the horizon problem.

2.1.5 The flatness problem

It is an observational problem associated with FRW metric [10]. In general, the universe can have three kinds of geometries (hyperbolic, Euclidean, or elliptic geometry) depending on the total energy density of the universe. It is hyperbolic if its density is less than the critical density, elliptic if greater, and Euclidean at the critical density. The critical density is the boundary value between the model of the universe which states that the universe will expand forever (open model) and the model which says that the universe will recollapse (closed model). A measurement of the actual density of the universe could be compared to the critical density in order to determine the fate of the universe. The critical density is given by [11]:

$$\rho_{crit} = \frac{3H_0^2}{8\pi G} = 5\times 10^{-29}\, g/cm^3 \tag{2.2}$$

where G is the gravitational constant. The curvature of space depends on the ratio $\Omega_0 = \frac{\rho}{\rho_{crit}}$. For Ω_0 greater than 1, the universe has positively curved or spherical geometry. For Ω_0 less than 1, the universe has negatively curved or hyperbolic geometry. For Ω_0 equal to 1, the universe has Euclidean or flat geometry. The behavior of the universe is determined according to its geometry.

The flatness problem arises because of the observation that the density of the universe today is very close to the critical density required for spatial flatness. Since the total energy density of the universe departs rapidly from the critical value over cosmic time, the early universe must have had a density even closer to the critical density, leading cosmologists to question how the density of the early universe came to be fine-tuned to this special value. In the later discussion, we will find cosmological equations that describe the universe in the three cases and for different kinds of matter. Figure (2.1) illustrates the kinds of geometries of the universe.

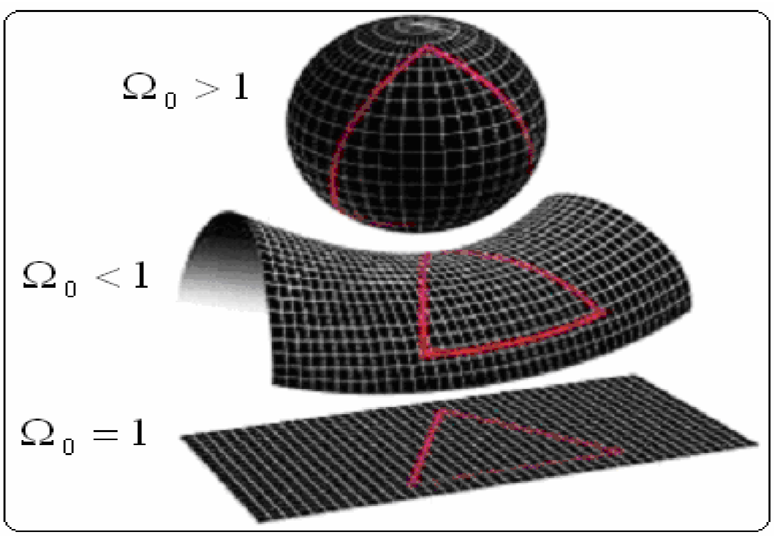

Figure (2.1) geometries of the universe [12]

2.1.6 De Sitter geometry

De Sitter geometry is the maximally symmetric nonsingular vacuum solution of Einstein's field equation with a positive cosmological constant Λ. This geometry describes the expansion of the universe according to the following equation:

$$a(t) = \cosh(t) \tag{2.3}$$

where $a(t)$ is the scale factor which describes the expansion of physical spatial distances and t is time. The vacuum dominated space is known as a de Sitter space [10]. Figure (2.2) illustrates the difference between the de sitter solution and a singular solution.

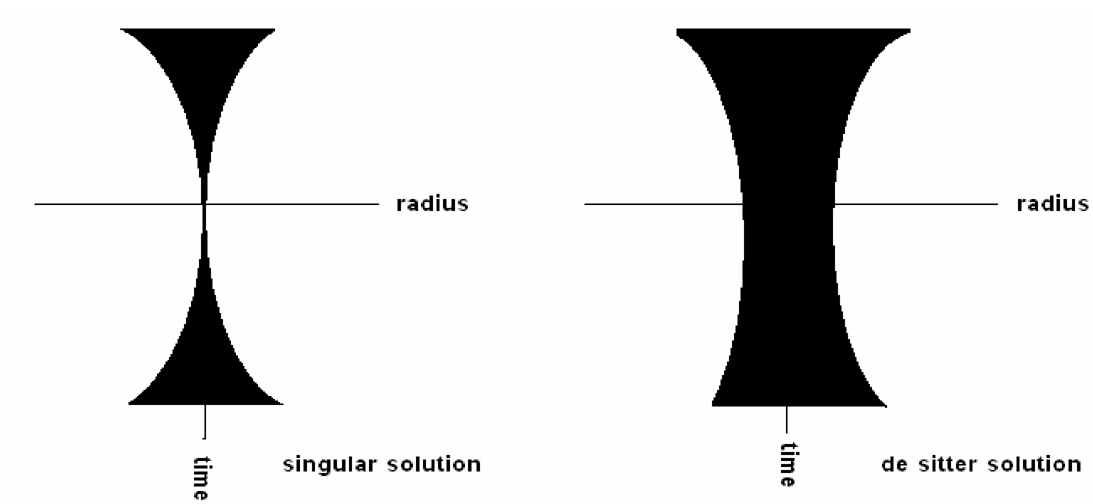

Figure (2.2) de sitter solution vs a singular solution.

2.1.7 Homogeneous and isotropic cosmology

Carroll [11] defines homogeneity as the statement that the metric is the same throughout the space. And he defines isotropy as the statement that at any specific point in the space, the space looks the same no matter what direction you look in. A space can be homogeneous but nowhere isotropic, or it can be isotropic around one point without being homogeneous (such as a cone, which is isotropic around its vertex but certainly not homogeneous). On the other hand, if a space is isotropic everywhere then it is homogeneous. Likewise, if it is isotropic around one point and also homogeneous, it will be isotropic around every point.

2.2 A brief review of CGR

It is well known, in classical mechanics, that gravitation is caused by forces between masses. In the theory of general relativity, Einstein showed that, instead, gravitation is due to curvature of spacetime that is caused by the presence of matter. The matter tends to pull the coordinates system toward them so the coordinates system appears to be curved (Figure (2.3)). CGR calls for the curvature of spacetime to be produced by the mass-energy and momentum content of the matter in spacetime. In the following subsections, we will discuss some fundamental topics in general relativity.

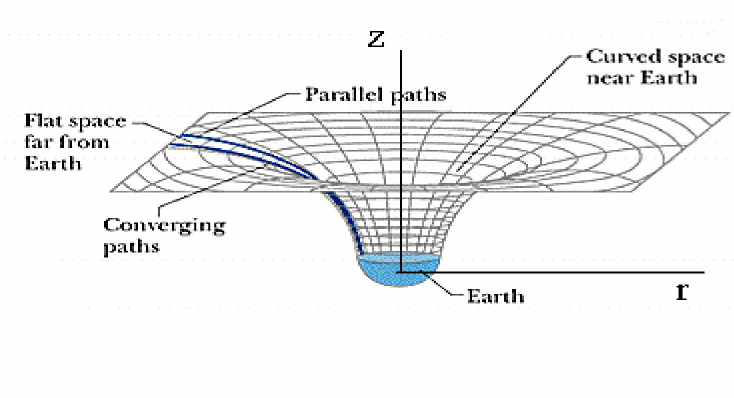

Figure (2.3) curvature caused by matter [13]

2.2.1 Principle of equivalence

“Einstein suddenly realized, while sitting in his office in Bern, Switzerland, in 1907, that if he were to fall freely in a gravitational field (think of a sky diver before she opens her parachute, or an unfortunate elevator if its cable breaks); he would be unable to feel his own weight. Einstein later recounted that this realization was the "happiest moment in his life", for he understood that this idea was the key to how to extend the Special Theory of Relativity to include the effect of gravitation. We are used to seeing astronauts in free fall as their spacecraft circles the Earth these days, but we should appreciate that in 1907 this was a rather remarkable insight”. This principle is the fundamental postulate of the general theory of relativity which says, in other words, that [13] “gravitation and acceleration are equivalent. If someone were locked up in a small box, he would not be able to tell whether the box was at rest on

Earth and subject only to the Earth's gravitational force, or accelerating through space at $9.8 m/s^2$ and subject only to the force producing that acceleration. In both situations he would feel the same and would read the same value for his weight on a scale. Moreover, if he watched an object fall past him, the object would have the same acceleration relative to him in both situations".

2.2.2 Manifolds and geodesics

A *manifold* is an abstract mathematical space in which every point has a neighborhood which resembles Euclidean space, but in which the global structure may be more complicated. It is the curved-space generalization of the notion of "Euclidean space". Carroll [11] said that the manifold is a space which may be curved and have a complicated topology, but in local regions it looks just like $\mathbf{R}^n$ which is the vector space for the n-dimensional vectors.

A *geodesic* is the curved-space generalization of the notion of a "straight line" in Euclidean space. We all know what a straight line is: it is the path of shortest distance between two points. But there is an equally good definition; a straight line is a path which parallel transports its own tangent vector.

2.2.3 Tensors

Tensors provide the mathematical framework for formulating and solving problems in areas of physics such as elasticity, fluid mechanics, and general relativity. Tensors are very important in general relativity since Einstein's field equations are tensor equations. In order to understand the mathematics of general relativity, we should understand the mathematics of tensors. In this subsection, we will talk about this important topic.

It is easy to understand the second-rank tensor as it is a matrix with a number of rows and columns. An n-th-rank tensor in m-dimensional space is a mathematical object that has n indices and m^n components and obeys certain transformation rules. Each index of a tensor ranges over the number of dimensions of space. Tensors are generalizations of scalars (that have no indices), vectors (that have exactly one index), and matrices (that have exactly two indices) to an arbitrary number of indices.

The notation for a tensor is similar to that of a matrix (i.e., a_{ij}), except that a tensor may have an arbitrary number of indices, a_{ijk}, a^{ijk},$a_i{}^{jk}$, where the upper indices are called "contravariant" indices and the lower indices are called "covariant" indices. Note that the positions of the slots in which contravariant and covariant indices are placed are significant so, for example, $a_{ij}{}^{k}$ is distinct from $a_i{}^{jk}$.

While the distinction between covariant and contravariant indices must be made for general tensors, the two are equivalent for tensors in three-dimensional Euclidean space, and such tensors are known as Cartesian tensors. The contraction of a tensor occurs when a pair of indices (one a subscript, the other a superscript) of the tensor are set equal to each other and summed over. In the Einstein notation this summation is built into the notation. The result is another tensor with rank reduced by two. The Einstein notation is: $a_i a^i = \sum_i a_i a^i$

The zeroth-rank tensors can be represented by scalars, first-rank tensors can be represented by vectors, and the second-rank tensors can be represented by matrices. In the following subsections, we will talk about four tensors that are very important in CGR, these tensors are: the metric tensor, the Riemann tensor, the Ricci tensor, and the stress-energy tensor.

2.2.3.1 The metric tensor

It is the fundamental object of study in general relativity. Mathematically, spacetime is represented by a 4-dimensional differentiable manifold M and the metric is given as a covariant second-rank symmetric tensor on M. Physicists usually work in local coordinates x^{μ} (where μ runs from 0 to 3):

$$x^0 = t, x^1 = x, x^2 = y, x^3 = z \tag{2.4}$$

the metric is represented by the following equation:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2.5}$$

the factors dx^μ are the gradients of the scalar coordinate fields x^μ. The metric is thus a linear combination of tensor products of the gradients of coordinates. With the quantity dx^μ being an infinitesimal coordinate displacement, the metric acts as an infinitesimal invariant interval squared or line element.

A simple example of the metric is the metric of flat spacetime which is [8]:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{2.6}$$

where $c = 1$. If we compare equation (2.5) with equation (2.6) taking into consideration equation (2.4), the metric tensor can be represented by the following matrix:

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.7}$$

From the matrix representation of the metric tensor we see that: $g^{00} = g^{tt} = -1$ (which is called the time-time component of the metric tensor, and $g^{11} = g^{22} = g^{33} = g^{rr} = 1$ (which are called the space-space components of the metric tensor).

In spherical coordinates, the flat space metric is:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \tag{2.8}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the standard metric on a 2-sphere which is considered as a good example of a space with curvature. It is the locus of points in R^2 at distance 1 from the origin.

Another metric is FRW metric which is [14,15,16]:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \tag{2.9}$$

here r is dimensionless and the dimension in a, k is a constant parameter that determines the curvature of the universe. If k=1, the universe is closed, positively curved, and finite. If k=0, the universe is open, flat, and infinite. If k=-1, the universe is open, negatively curved, and infinite. FRW is the standard big bang model. It is the solution of the gravitational field equations of general relativity. These can describe open or closed universes. All these FRW universes have a singularity at the origin of time which represents the big bang. FRW spacetimes come in a great variety of styles, expanding, contracting, flat, curved, open, closed, etc. The relation between Ω_0 and k is [11]:

$$\Omega_0 - 1 = \frac{k}{H_0^2 a^2} \tag{2.10}$$

Also there is an important metric in CGR which is the Schwarzschild metric [11,18]:

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \qquad (2.11)$$

where M is a constant with the dimensions of mass. The Schwarzschild solution (we will derive it later) describes the gravitational field outside a spherical, non-rotating mass such as a (non-rotating) star or black hole. It is also a good approximation to the gravitational field of a slowly rotating body like the Earth or Sun. It is the most general spherically symmetric, vacuum solution of the Einstein field equations.

2.2.3.2 Riemann tensor, Ricci tensor, and Ricci scalar

The Riemann tensor [19], or the Riemann-Christoffel curvature tensor [17], or Riemann curvature tensor [14] is a four-index tensor that is useful in general relativity since it gives a description of the curvature of spacetime. It is the only fourth-rank tensor that can be constructed from the metric tensor and its first and second derivatives. Since the Riemann tensor is a four-index tensor in a four-dimensional spacetime, it has a 4^4 = 256 components. The Riemann tensor is defined in the following equation [20]:

$$R^{\mu}_{\ \nu\alpha\beta} = \Gamma^{\mu}_{\nu\beta,\alpha} - \Gamma^{\mu}_{\nu\alpha,\beta} + \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\sigma}_{\nu\beta} - \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\nu\alpha} \qquad (2.12)$$

where Γ are the Christoffel symbols and the comma in the equation denotes differentiation ($\Gamma^{\mu}_{\nu\beta,\alpha} = \partial_{\alpha}\Gamma^{\mu}_{\nu\beta}$). Christoffel symbols are defined from the metric tensor and its derivatives according to the following equation [11]:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\rho\mu} - \partial_{\rho} g_{\mu\nu}\right) \tag{2.13}$$

The contraction of the Riemann tensor on the first and third indices is known as the Ricci tensor [11]:

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} \tag{2.14}$$

the components of the Ricci tensor for metric (2.9) are [17]:

$$\begin{aligned} R_{tt} &= 3\frac{\ddot{a}}{a} \\ R_{ij} &= -(a\ddot{a} + 2\dot{a}^2 + 2k)\tilde{g}_{ij} \end{aligned} \tag{2.15}$$

where R_{tt} is the time-time component of the Ricci tensor, R_{ij} is the space-space components, and $\tilde{g}_{ij}$ is the metric for a three-dimensional maximally symmetric space which can be defined according to the following equation:

$$g_{ij} = a^2(t)\tilde{g}_{ij} \tag{2.16}$$

The contraction of the Ricci tensor is the Ricci scalar (also known as the scalar curvature). It is obtained by setting the indices of the Ricci tensor equal [11]:

$$R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu} \tag{2.17}$$

If we use Einstein notation here we have $R = R^{\mu}_{\mu} = \sum_{\mu} R^{\mu}_{\mu} = R^{0}_{0} + R^{1}_{1} + \ldots$, this summation is equal to the trace of the Ricci tensor. The relation between the Ricci scalar and the scale factor is obtained by multiplying the metric tensor matrix with the Ricci tensor matrix and finding the trace of the resulted matrix. This procedure gives:

$$R = -\frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + k) \tag{2.18}$$

2.2.3.3 Stress-energy tensor

In this section, we will define the stress-energy tensor or the energy-momentum tensor of a perfect fluid. It tells us all we need to know about the system as energy density, pressure, stress, etc. Let's consider the very general category of matter which may be characterized as a fluid (a continuum of matter described by macroscopic quantities such as temperature, pressure, viscosity, etc. In general relativity, all interesting types of matter can be thought of as *perfect fluids*, from stars to electromagnetic fields to the entire universe. Schutz [19] defines a perfect fluid to be one with no heat conduction and no viscosity, while Weinberg [17] defines it as a fluid which looks isotropic in its reference frame and these two definitions seem to be equivalent. Operationally, you should think of a perfect fluid as one which may be completely characterized by its pressure and density. The stress-energy tensor of a perfect fluid is defined as [10,11]:

$$T^{\mu\nu} = (\rho + p)U^{\mu}U^{\nu} + pg^{\mu\nu} \tag{2.19}$$

where ρ is the fluid's density, p is its pressure, and U is the 4-velocity of the fluid.

The 4-velocity is a four-vector, a vector in four-dimensional spacetime that replaces classical velocity, a three-dimensional vector. The components of the 4-velocity of a perfect fluid at rest or in comoving frames are given by:

$$\begin{aligned} U^{t} &= 1 \\ U^{i} &= 0 \end{aligned} \qquad (2.20).$$

Figure (2.4) shows the components of the stress-energy tensor.

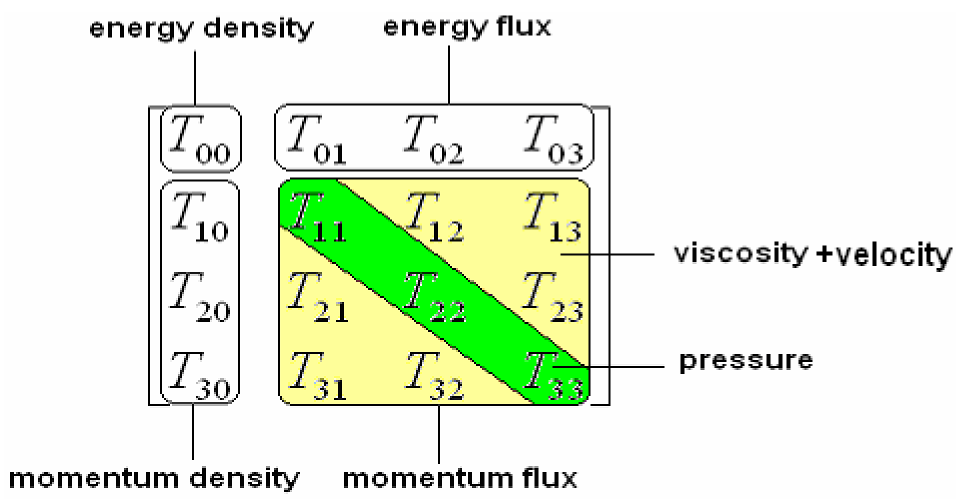

Figure (2.4) components of the stress-energy tensor [21]

By using equations (2.7), (2.19), (2.20) we can write the stress-energy tensor of a perfect fluid at rest in matrix form as:

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{2.21}$$

as we can notice the elements ($T^{ij}, i \neq j$) are all zero since there are neither viscosity nor heat conduction nor motion in the perfect fluid. From the matrix representation of the stress-energy tensor we see that: $T^{00} = T^{tt} = \rho$ (the time-time component of the stress-energy tensor), and $T^{11} = T^{22} = T^{33} = T^{rr} = p$ (the space-space components of the stress-energy tensor).

2.2.4 Einstein field equations (EFEs)

EFEs can be derived from the Hilbert-Einstein action [22]:

$$S_g = h\int R\sqrt{-g}\,d^4x \tag{2.22}$$

where g is the determinant of the metric tensor ($g^{\mu\nu}$), R is the Ricci scalar curvature which is the trace of the Ricci curvature tensor ($R^{\mu\nu}$), h is the constant *1/16πG*, $R\sqrt{-g}$ is the Lagrangian density, and the integral is taken over a specified region of spacetime.

To derive the full field equations, a matter Lagrangian L_M is added to the above action:

$$S_{tot} = \int [hR + L_M]\sqrt{-g}\,d^4x \tag{2.23}$$

In physics, the action is an integral quantity that is used to determine the evolution of a physical system between two defined states. The evolution of a physical system between two states is determined by requiring the action to be minimized or, more generally, to be stationary. This requirement leads to differential equations that describe the true evolution. Conversely, an action principle is a method for reformulating differential equations of motion for a physical system as an equivalent integral equation.

The most commonly used action principle is Hamilton's principle which states that the true evolution $q(t)$ of a system described by N generalized coordinates $q = (q_1, q_2, \ldots, q_N)$ between two specified states $q(t_1)$ and $q(t_2)$ at two specified times t_1, t_2 is an extremum (i.e., a stationary point, a minimum, a maximum, or a saddle point) of the action $S[q(t)] = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt$ where $L(q, \dot{q}, t)$ is the Lagrangian function of the system. Accordingly, the actual evolution of a physical system is the solution of the equation:

$$\frac{\delta S}{\delta q(t)} = 0. \qquad (2.24)$$

Let's go to an analogy from one-dimensional classical mechanics by taking the following action:

$$S = \int_{t_1}^{t_2} L(x, \dot{x}) dt \tag{2.25}$$

we assume here that the Lagrangian L (the integrand of the action integral) depends only on the coordinate x and its time derivative $\dot{x}$, and does not depend on time explicitly. The requirement that S be stationary implies that δS must be zero, this can be true only if :

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \tag{2.26}$$

which is the Euler-Lagrange equation.

The EFEs may be obtained from the variation of the action in equation (2.23) with respect to $g_{\mu\nu}$:

$$\frac{\delta S_{tot}}{\delta g_{\mu\nu}} = \int \left[h\left(\frac{\delta R}{\delta g_{\mu\nu}} + R\frac{\delta\sqrt{-g}}{\sqrt{-g}\delta g_{\mu\nu}} \right) + \frac{\delta\left(\sqrt{-g}L_M\right)}{\sqrt{-g}\delta g_{\mu\nu}} \right] \sqrt{-g}d^4x \tag{2.27}$$

now, we have the following equations [22]:

$$\frac{\delta S_{tot}}{\delta g_{\mu\nu}} = 0 \tag{2.28}$$

which is the condition for the actual evolution of the physical system.

$$\frac{\delta R}{\delta g_{\mu\nu}} = R^{\mu\nu} \tag{2.29}$$

$$\frac{\delta\sqrt{-g}}{\sqrt{-g}\delta g_{\mu\nu}} = -\frac{1}{2}g^{\mu\nu} \tag{2.30}$$

since the stress-energy tensor describes the matter, it must be derived from a matter lagrangian:

$$\frac{\delta(\sqrt{-g}L_M)}{\sqrt{-g}\delta g_{\mu\nu}} = -\frac{1}{2}T^{\mu\nu}. \tag{2.31}$$

By substituting into equation (2.27), we get:

$$0 = \int\left[h\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right) - \frac{1}{2}T^{\mu\nu}\right](\delta g_{\mu\nu})\sqrt{-g}d^4x \tag{2.32}$$

the expression in brackets must be zero, so:

$$h\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right) - \frac{1}{2}T^{\mu\nu} = 0 \tag{2.33}$$

and

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = 8\pi GT^{\mu\nu} \tag{2.34}$$

which is (EFEs). The above form of the EFEs is for the +- - - metric sign convention, which is commonly used in general relativity. Using the -+++ metric sign convention, which is used in this work, leads to an alternate form of the EFEs which is [17,24,25]:

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = -8\pi GT^{\mu\nu} \tag{2.35}$$

2.3 The Limiting Curvature Hypothesis (LCH)

As we discussed so far, the singularity theorems of Hawking and Penrose imply that general relativity gives an incomplete description of the behavior of spacetime at high curvatures. So, we must implement a more realistic cosmological model that gives a more comprehensive description of the behavior of spacetime at high curvatures. This can be done by modifying EFEs by creating new field equations that achieve our goals. On the other hand, our new field equations must approach Einstein's field equations at low curvatures where Einstein's theory of general relativity is working properly. Now, let's talk about Penrose diagrams [23], which give a clear explanation of the limiting curvature hypothesis. Figure (2.5) shows these diagrams which are talking about the collapsing universe (left) and black holes (right) in Einstein's theory (top) and in the nonsingular universe (bottom). C, E, DS and H stand for contracting phase, expanding phase, de Sitter phase and horizon, respectively, and wavy lines indicate singularities.

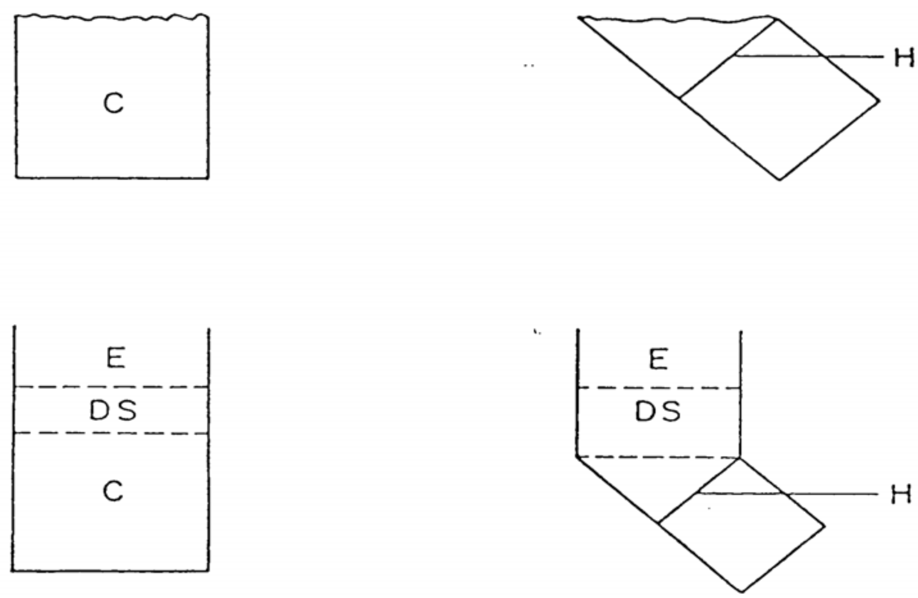

Figure (2.5) Penrose diagrams

If successful, the above construction will have some consequences. Consider, for example, a collapsing spatially homogeneous universe. According to Einstein's theory, this universe will collapse in finite proper time to a final "big crunch" singularity (top left Penrose diagram). In our theory, however, the universe will approach a de Sitter model as the curvature increases. If the universe is closed, there will be a de Sitter bounce followed by re-expansion (bottom left Penrose diagram). Similarly, in our theory spherically symmetric vacuum solutions would be nonsingular, i.e., black holes would have no singularities in their centers (bottom right) compared to what is predicted by Einstein's theory (top right) which says that a black hole will collapse into a singularity.

2.3.1 Implementing LCH

In order to implement (LCH), we must modify Hilbert-Einstein action by putting a limit to the Ricci scalar, which is the quantity to be limited . To illustrate this, let's go to the analogy [8,23] with the action for point particle motion in special relativity which can be obtained from Newtonian mechanics. The starting point is the Newtonian action for a point particle:

$$S = \int L(x, \dot{x}, t) dt = m \int dt \frac{1}{2} \dot{x}^2 \tag{2.36}$$

In classical theory, there is no bound on the velocity. So, this action must be modified to give a bound on the velocity. By adding a Lagrange multiplier φ which couples to $\dot{x}^2$, the scalar quantity which is to be limited, and giving φ a potential $V(\varphi)$ the new action is:

$$S = m \int dt \left[\frac{1}{2} \dot{x}^2 + \varphi \dot{x}^2 - V(\varphi) \right] \tag{2.37}$$

provided that $V(\varphi) \sim \varphi$ for $|\varphi| \to \infty$. The constraint equation ensures that $\dot{x}$ is bounded.

In order to obtain the correct Newtonian limit for small velocities (i.e., small φ), $V(\varphi)$ must be proportional to φ^2 for $|\varphi| \to 0$. As a result, the simplest potential which satisfies the above conditions is:

$$V(\varphi)=\frac{2\varphi^2}{1+2\varphi} \tag{2.38}$$

Eliminating the Lagrange multiplier using the constraint equation and substituting the result into the action yields the point particle action in special relativity:

$$S=m\int dt\sqrt{1-\dot{x}^2} \tag{2.39}$$

So, to implement (LCH), we must modify the Hilbert-Einstein action and postulate an action like [8,23]:

$$S_g=k\int\left[R+\varphi_1 R+V_1(\varphi_1)\right]\sqrt{-g}\,d^4x \tag{2.40}$$

Let us consider $L(R)=R+\varphi_1 R+V_1(\varphi_1)$. At low curvature, *L(R)* must approach *R* so that our action approaches the Hilbert-Einstein action. On the other hand, in the region of space where the curvature approaches the limiting value, our action approaches the Hilbert-Einstein action with a certain cosmological constant.

2.3.2 The new field equations

The field equations corresponding to the above action in equation (2.40) can be obtained from the variation of the action for matter plus gravity with respect to $g_{\mu\nu}$, the same way we derived EFEs. To derive the full field equations, a matter Lagrangian L_M must be added to the action:

$$S_g = \int (kL(R) + L_M)\sqrt{-g}\,d^4x \tag{2.41}$$

Now, the problem is to find an expression for *L(R)* (this will be left for a future work) so that the variation of the action with respect to $g_{\mu\nu}$ yields the following field equations [6]:

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} - \frac{1}{4}\Lambda(1-U)g^{\mu\nu} = -8\pi G T^{\mu\nu} \tag{2.42}$$

where

$$U = \sqrt{1 - \frac{R^2}{\Lambda^2}} \tag{2.43}$$

and Λ is a limiting curvature scale and a cosmological constant at the same time. We restrict R to the range $-\Lambda \le R \le \Lambda$ and $\Lambda \ge 0$. My subsequent work will be based on these field equations.

We notice that equation (2.42) is a modification of Einstein's field equations (2.35). We modify Einstein's field equations by inserting a certain cosmological constant in it. As we can see, equation (2.42) approaches the Einstein field equation at low curvatures (when $R \to 0$).

By returning to equation (2.42), it is obvious that there is a limiting value of curvature $|R| = \Lambda$ beyond which the equation become imaginary leading to a manifest implementation of the limiting curvature hypothesis the same way as implementing the limiting speed postulate in special relativity.

In the following section we will find solutions of the field equation for different kinds of matter beginning with the limiting state case.

2.3.3 Finding cosmological equations

By contracting the field equation (2.42) with respect to the μ and ν indices we get the trace equation:

$$-R-\Lambda(1-U)=-8\pi GT \tag{2.44}$$

or

$$\frac{R}{\Lambda}+1-U=\frac{8\pi GT}{\Lambda} \tag{2.45}$$

where T is the trace of T_{ν}^{μ}. Now, we shall introduce the following notations: $\beta=\dfrac{R}{\Lambda}$ and $\gamma=\dfrac{8\pi GT}{\Lambda}$. Equation (2.45) becomes:

$$\beta+1-\sqrt{1-\beta^{2}}=\gamma \tag{2.46}$$

this equation yields:

$$2\beta^{2}+2(1-\gamma)\beta+\gamma^{2}-2\gamma=0\,. \tag{2.47}$$

This equation has two solutions:

$$\beta=-\frac{1}{2}(1-\gamma)\pm\frac{1}{2}\sqrt{1-\gamma^{2}+2\gamma} \tag{2.48}$$

The next step is to find differential equations from the field equation that describe the relation between the scale factor (*a*), the radius of the universe, and time for different kinds of matter.

2.3.3.1 The limiting state (vacuum dominated universe)

LCH is constructed according to two important points [8]. The first one is that all curvature invariants are bounded, and the second is that when these invariants approach their limiting value, a nonsingular de Sitter solution is taken on. So, at the limiting state we will have a de Sitter space and thus a vacuum dominated universe. As a result, the stress-energy tensor vanishes because there is neither pressure nor energy density in vacuum. So, $T = 0$ and hence $\gamma = 0$. Now according to equation (2.48) the value of β corresponding to $\gamma = 0$ is either 0 or -1, and since we study the limiting state case, we choose β = -1. Now, when $\gamma = 0$ and $\beta = -1$ the field equation (2.42) becomes:

$$R^{\mu\nu} + \frac{1}{4}\Lambda g^{\mu\nu} = 0 \tag{2.49}$$

and the time-time component of the field equations takes the form [17]:

$$3\frac{\ddot{a}}{a} - \frac{1}{4}\Lambda = 0 \tag{2.50}$$

and the space-space components are [17]:

$$\frac{1}{4}\Lambda - \frac{1}{a^2}\left(a\ddot{a} + 2\dot{a}^2 + 2k\right) = 0 \tag{2.51}$$

where k was discussed so far. We can combine equations (2.50) and (2.51) to get a first order differential equation:

$$\dot{a}^2 = \frac{\Lambda a^2}{12} - k \tag{2.52}$$

also we can get a second order differential equation from these equations by adding them:

$$\ddot{a} - \frac{\dot{a}^2 + k}{a} = 0 \tag{2.53}$$

2.3.3.2 The radiation dominated universe

The big bang starts off with a state of extremely high density and pressure for the universe. Under those conditions, the universe is dominated by radiation. This means that the majority of the energy is in the form of photons and other massless or nearly massless particles (like neutrinos) that move at near the speed of light. As the big bang evolves in time, the temperature drops rapidly as the universe expands and the average velocity of particles decreases.

The existence of the Cosmic Microwave Background radiation (CMB) suggests that the universe was governed by radiation for most of the first 100,000 years until the energy density of matter became larger than that of radiation such that the energy of the matter began to dominate the universe's evolution [26]. In that epoch, the radiation density or the photons density decreases just like the matter density so it goes as:

$$\text{Photon density } = \frac{N}{a^3} \tag{2.54}$$

where N is the number of photons, and a is the scale factor. But not only does the photons density decrease with time, the average energy per photon also decreases because the universe is expanding and cooling therefore we have the following:

$$\text{Photon energy density} = \frac{N}{a^3}\frac{1}{a} = \frac{N}{a^4} \tag{2.55}$$

the extra term $\frac{1}{a}$ represents the average energy per photon. In this case, the equation of state becomes that of pure radiation [11,17,27]:

$$p = \frac{\rho}{3} \tag{2.56}$$

where p is the pressure and ρ is the energy density. The trace of the stress-energy tensor vanishes since $T = 3p - \rho$, as a result $\gamma = 0$. We choose β = -1 since we want to choose β = 0 for the empty flat spacetime. The field equations (2.42) become (for $\gamma = 0$ and $\beta = -1$):

$$R^{\mu\nu} + \frac{1}{4}\Lambda g^{\mu\nu} = -8\pi G T^{\mu\nu} \tag{2.57}$$

the time-time component of the field equations becomes:

$$3\frac{\ddot{a}}{a} - \frac{1}{4}\Lambda = -8\pi G\rho \tag{2.58}$$

while the space-space component of the field equations gives:

$$\frac{1}{4}\Lambda - \frac{1}{a^2}\left(a\ddot{a} + 2\dot{a}^2 + 2k\right) = -8\pi Gp \tag{2.59}$$

Equations (2.58) and (2.59) may be combined to eliminate $\ddot{a}$ and this yields:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + \frac{\Lambda}{12}a^2 - k \tag{2.60}$$

According to equation (2.55) ρ depends on the scale factor so ρ can be written as [6]:

$$\rho = \rho_r = \rho_r^0 \frac{a_0^4}{a^4} \tag{2.61}$$

where ρ_r^0 and a_0 are the present values of the radiation density and the scale factor.

Finally, we will introduce the relation between the scale factor and time in a radiation dominated universe as suggested by CGR for k = 0, which is a singular solution [26];

$$a(t) = bt^{1/2} \tag{2.62}$$

where b is a constant.

2.3.3.3 The matter dominated universe

The energy density of the known forms of radiation in the present universe is less than one-hundredth the density of the rest-mass. In other words, we have reached a state where the energy of the universe is primarily contained in non-relativistic matter (matter sufficiently massive that its average velocity is very much less than the speed of light and the pressure generated there is extremely small compared with the energy density). This is called a *matter dominated universe*. The early universe was radiation dominated, but the present universe is matter dominated. In the matter dominated universe, which is the present epoch, the main energy density is that of ordinary matter in galaxies, whose random velocities are small and which therefore behave like dust. So, we deal with non-relativistic matter particles, and the pressure is zero. As a result, the trace of the energy-momentum tensor is equal to $(-\rho)$ and $\gamma = \frac{-8\pi G\rho}{\Lambda}$. Therefore, γ is a function of time because ρ is a function of time and so is β because according to equation (2.48) it depends on γ.

The field equations (2.42) can be rewritten as:

$$R^{\mu\nu} + f(t)\Lambda g^{\mu\nu} = -8\pi G T^{\mu\nu} \tag{2.63}$$

where $f(t) = -\frac{1}{4}(2\beta + 1 - \sqrt{1-\beta^2})$ and $\beta = \frac{R}{\Lambda}$. The time-time component of equation (2.63) is:

$$3\frac{\ddot{a}}{a} - f(t)\Lambda = -8\pi G\rho \tag{2.64}$$

and the space-space components are:

$$f(t)\Lambda - \frac{1}{a^2}\left(a\ddot{a} + 2\dot{a}^2 + 2k\right) = 0 \tag{2.65}$$

we can use equations (2.64) and (2.65) to eliminate the second derivative of the scale factor and obtain a first order differential equation:

$$\dot{a}^2 = \frac{8\pi G}{6}\rho a^2 + \frac{1}{3}f(t)\Lambda a^2 - k \tag{2.66}$$

We can get another second order differential equation by adding equations (2.64) and (2.65):

$$a\ddot{a} - \dot{a}^2 = k - 4\pi G\rho a^2 \tag{2.67}$$

The density in the matter dominated universe is equal to the mass of the universe divided by its volume, and since the volume is proportional to the cube of the radius, then matter density depends on the scale factor according to the following equation:

$$\rho = \frac{b}{a^3} \tag{2.68}$$

where b is a constant which is approximately equal to the mass of the present universe. However one can get a second independent equation from the equations (2.18), (2.48), and by using the following equation:

$$R = \beta\Lambda \tag{2.69}$$

this yields:

$$a\ddot{a}+\dot{a}^2+k=\frac{\Lambda a^2}{12}\left(1-\gamma\pm\sqrt{1-\gamma^2+2\gamma}\right) \tag{2.70}$$

If we combine equations (2.67) and (2.70) to eliminate the second order derivative term we get:

$$\dot{a}^2+k=2\pi G\rho a^2+\frac{\Lambda a^2}{24}\left(1-\gamma\pm\sqrt{1-\gamma^2+2\gamma}\right) \tag{2.71}$$

It is also possible to get another first order equation by integrating equation (2.67), and we get:

$$\dot{a}^2=Ca^2-k-8\pi Ga^2I(a) \tag{2.72}$$

where $I=\int\frac{\rho}{a}da$ and C is a constant that can be evaluated from the present values of the Hubble constant, matter density, and scale factor;

$$C=H_0^2-\frac{8\pi G}{3}\rho_0+\frac{k}{a_0^2}\,. \tag{2.73}$$

Another way to proceed is to find an expression for γ in terms of the scale factor and its derivatives from equations (2.70) and substitute it in the trace equation after having removed the square root, this yields:

$$\begin{aligned}&25(a\ddot{a})^2+34a\ddot{a}\left(\dot{a}^2+k\right)+13\left(\dot{a}^2+k\right)^2\\&-2\Lambda a^2\left(2a\ddot{a}+\dot{a}^2+k\right)=0\end{aligned} \tag{2.74}$$

If we put $Y=\dot{a}^2+k$ in equation (2.74) then $\frac{dY}{da}=2\ddot{a}$. Equation (2.74) becomes:

$$\frac{25}{4\Lambda a^2}\left(\frac{dY}{da}\right)^2 - \frac{2}{a}\frac{dY}{da} + \frac{17}{\Lambda a^3}Y\frac{dY}{da} + \frac{13}{\Lambda a^4}Y^2 - \frac{2}{a^2}Y = 0 \qquad (2.75)$$

The scale factor in the matter dominated universe as a function of time for the three cases of k (k = +1, k = 0, k = -1) as predicted by CGR is shown in figure (2.6).

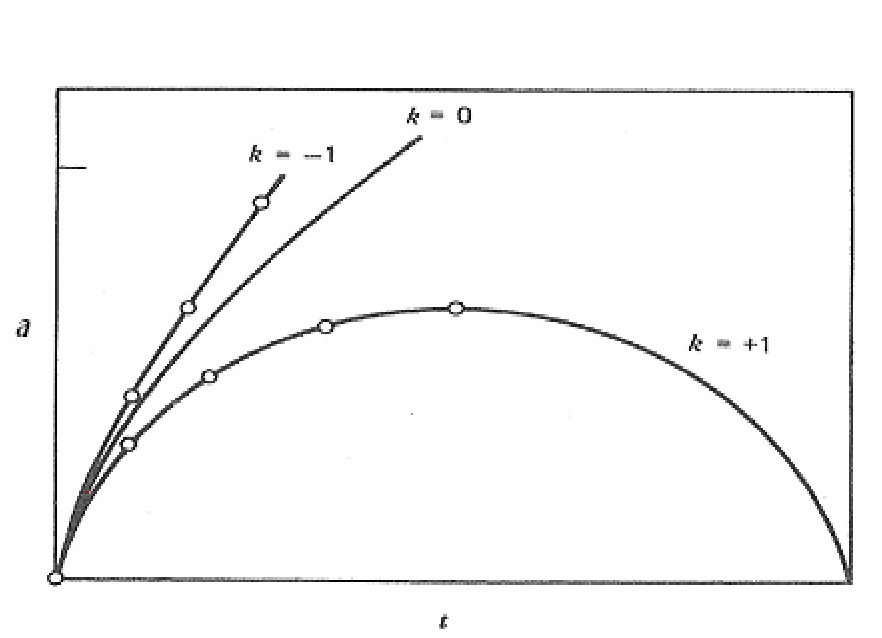

Figure (2.6) Scale factor vs time in the matter dominated universe [17]

2.3.3.4 Combination of matter and radiation

When the radiation density equals the matter density, the universe is neither radiation dominated nor matter dominated since the two densities are equal. In this case we can say that the universe is dominated by both matter and radiation, the matter in the form of galaxies and radiation being represented by the microwave background radiation. The field equations in this case take the form of (2.63). The time-time component of the field equations is:

$$3\frac{\ddot{a}}{a} - f(t)\Lambda = -8\pi G(\rho_m + \rho_r) \tag{2.76}$$

where ρ_m and ρ_r are the matter and radiation densities respectively. The space-space component is:

$$f(t)\Lambda - \frac{1}{a^2}\left(a\ddot{a} + 2\dot{a}^2 + 2k\right) = -8\pi G(p_m + p_r) \tag{2.77}$$

where p_m and p_r are the matter and radiation pressures respectively. The two equations can be combined to obtain a first order differential equation:

$$\dot{a}^2 = \frac{8\pi G}{6}(\rho_m + \rho_r)a^2 + \frac{1}{3}f(t)\Lambda a^2 - k \tag{2.78}$$

Now, let's look at the behavior of the solution of this differential equation: for values of the scale factor near the minimum radius (the minimum radius is obtained by setting the rate of expansion $\dot{a}$ equal to zero and k = 1 and solving for a), the universe is radiation dominated and the solution of (2.78) is the same as the solution of (2.60) which stops and bounces at the minimum radius. For large values of the scale factor the universe becomes matter dominated and reaches maximum radius and collapses. The minimum radius is not zero and the initial and final states of the universe are not singular.

2.3.3.5 Spherically symmetric solutions

In this section, we will obtain a solution of the field equations outside a spherically symmetric massive object, and a solution that represents a collapsing star (this solution will demonstrate the relation between the star radius and time when the star collapses and dies). After the formation of the star, the star lives the longest and most stable period in its life (approximately 10^9 years). In this period, it burns hydrogen in its core and converts it to helium, generating heat and light. In this period, the life of a star is a struggle between the inward pull of gravity and the outward push of pressure. The force of gravity comes from the attraction between the core of the star and the outer layers and the pressure comes from the heat produced by the burning of hydrogen. When the fuel is used up, the temperature declines and the star begins to shrink as gravity starts winning the struggle.

As the core of the star burns all the hydrogen into helium at the end its life, the star becomes a *red supergiant.* In this stage the core of the star shrinks, becoming hotter and denser, and the outer layers expand. After that, different nuclear processes occur like fusion which produces heavier elements that temporarily stop the core's shrinking. Eventually this core collapses (in an instant). As the iron atoms are crushed together in this gravitational collapse, the core temperature rises to about 100 billion degrees. The repulsive electrical forces between the atoms' nuclei overcome the gravitational forces, causing a massive, bright, short-lived explosion called a *supernova*. During this explosion, the star's outer layers are thrown away.

The next stage determines the fate of the star depending on the remaining mass of the star (the core). For the Sun-like Stars (mass under 1.5 times the mass of the Sun), they will collapse into a *white dwarf*. If the star's remaining mass is between 1.5 to 3 times the mass of the Sun, it will collapse into a *neutron star*. If the star's remaining mass is greater than three times the mass of the Sun, the star will collapse and become a *black hole* which is an incredibly dense body with a gravitational field so strong that even light cannot escape. It is a body in which all of the mass has collapsed gravitationally inside the point of possible escape. This point of no return, given by the surface $r = 2GM$, is known as the *event horizon*, and can be thought of as the surface of a black hole. The black hole is an object with a gravitational field so powerful that a region of space becomes cut off from the rest of the universe, no matter or radiation that has entered the region can ever escape. As not even light can escape, black holes appear black. When we talk about black holes we are not talking about something that is imaginary or does not exist in the real world. Cosmologists detected black holes in our galaxy which are not singular (i.e they have volume and their radii are not zero). For example, one of these black holes is known as *Cygnus X-1* which is located in *Cygnus* constellation.

Cygnus X-1 is a source located in our galaxy, the Milky Way, at a distance of about 10,000 light-years from the Earth. It is a binary system composed of a blue giant star 33 times more massive than the Sun and of an extremely dense and compact object of 15 solar masses. The compact object in the Cygnus X-1 binary is most probably a black hole. The picture in figure (2.7) is a composite image that shows the binary system Cygnus X-1. This illustration shows how matter from the giant blue star (left) is accreted in a spiraling disk of material around a black hole (right).

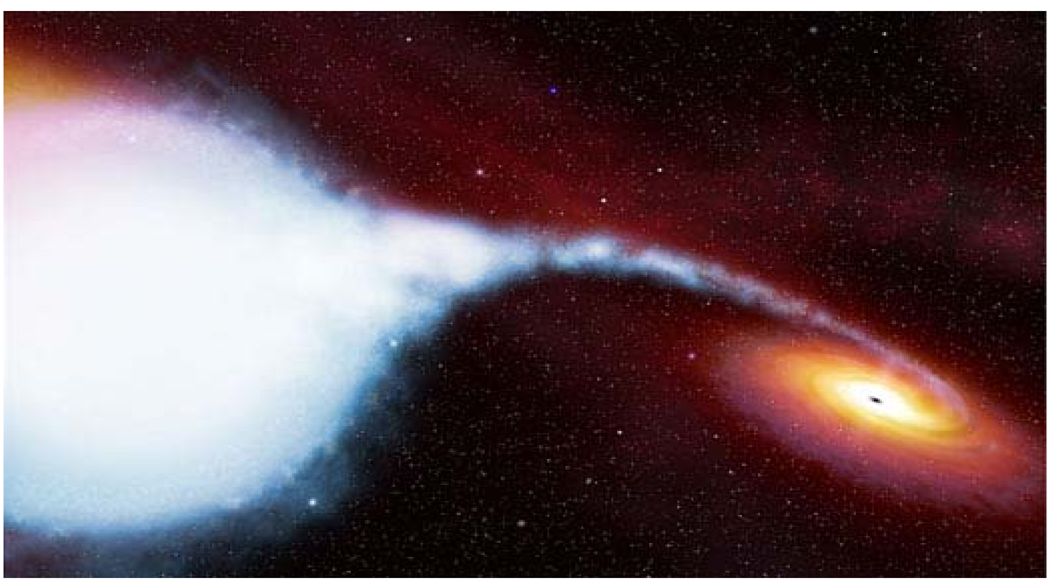

Figure (2.7) the binary system Cygnus X-1 [28]

The cosmologists determine the location of black holes by watching the effect of black holes on the coordinate system around them. They see that the coordinate system around the black holes is curved. Black holes as presently understood are described by the theory of general relativity. This theory showed that gravitation is due to curvature of space that is caused by the presence of masses and predicts that when a large enough amount of mass is present within a sufficiently small region of space, all paths through space are warped inwards towards the center of the volume. When an object is compressed enough for this to occur, collapse is unavoidable (it would take infinite force to resist collapsing into a black hole). When an object passes within the event horizon at the boundary of the black hole, it is lost forever (it would take an infinite amount of effort for an object to climb out from inside the hole). Figure (2.8) shows the different stages in the death of giant stars [29];

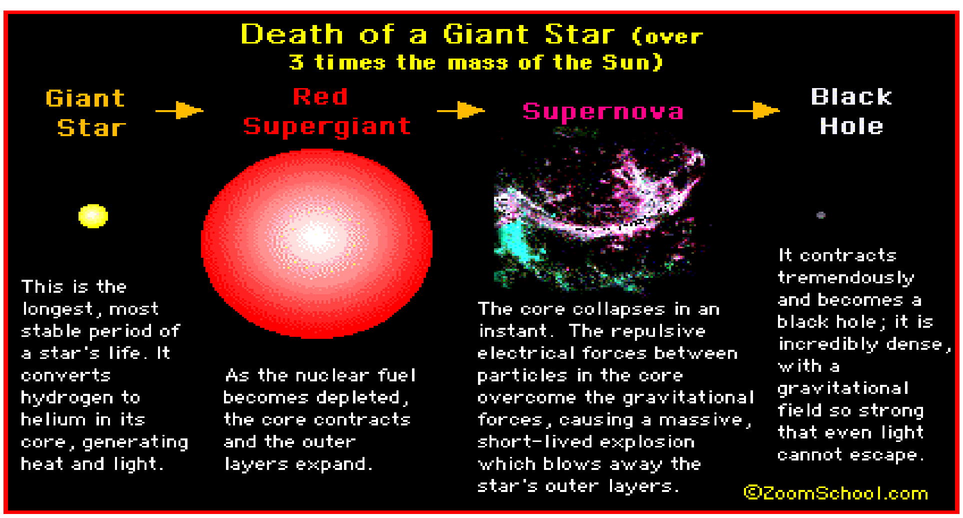

Figure (2.8) Death of giant stars

In vacuum flat spactime, the stress-energy tensor vanishes since there is no pressure and energy density. As a result, $\gamma = 0$ and β = 0 or -1. Since we choose β = -1 for the radiation dominated universe, we will choose $\beta = 0$ for the vacuum flat spacetime. By substituting $\gamma = \beta = 0$, the field equations (2.42) become:

$$R^{\mu\nu} = 0 \tag{2.79}$$

which are Einstein's field equations for the vacuum.

2.3.3.6 Derivation of the Schwarzschild solution

Now, let's return to equation (2.11), to derive this solution we begin from the standard form of the metric which is [17]:

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 \tag{2.80}$$

The Ricci tensor can be defined by using Christoffel symbols as [17]:

$$R_{\mu\nu} = \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} + \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\nu\eta} - \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\lambda\eta} \tag{2.81}$$

by using equations (2.13) and (2.81) we can find the components of the Ricci tensor:

$$R_{rr} = \frac{B''(r)}{2B(r)} - \frac{1}{4}\left(\frac{B'(r)}{B(r)}\right)\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{1}{r}\left(\frac{A'(r)}{A(r)}\right) \tag{2.82}$$

$$R_{\theta\theta} = -1 + \frac{r}{2A(r)}\left(-\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) + \frac{1}{A(r)}$$

$$R_{\varphi\varphi} = \sin^2\theta R_{\theta\theta}$$

$$R_{tt}=-\frac{B''(r)}{2A(r)}+\frac{1}{4}\left(\frac{B'(r)}{A(r)}\right)\left(\frac{A'(r)}{A(r)}+\frac{B'(r)}{B(r)}\right)-\frac{1}{r}\left(\frac{B'(r)}{A(r)}\right)$$

$R_{\mu\nu}=0$ for $\mu\neq\nu$

From equations (2.82) we have:

$$\frac{R_{rr}}{A}+\frac{R_{tt}}{B}=-\frac{1}{rA}\left(\frac{A'}{A}+\frac{B'}{B}\right) \tag{2.83}$$

The Schwarzschild solution represents the solution of EFEs for empty space so equation (2.79) applies here. As a result, we see that it will suffice to set the components of the Ricci tensor defined in equations (2.82) equal to zero, so equation (2.83) requires that $B'/B=-A'/A$ or:

$$A(r)B(r)=\text{Constant.} \tag{2.84}$$

For $r\to\infty$, the standard metric must approach the flat space metric in spherical coordinates (equation (2.8)), that is:

$$\lim_{r\to\infty}A(r)=\lim_{r\to\infty}B(r)=1 \tag{2.85}$$

from equations (2.84) and (2.85) we get:

$$A(r)=\frac{1}{B(r)} \tag{2.86}$$

by using (2.86) in (2.82) we get:

$$R_{\theta\theta}=-1+B'(r)r+B(r) \tag{2.87}$$

and since $R_{\theta\theta}=0$ this yield:

$$B'(r)r + B(r) = 1 \tag{2.88}$$

or

$$\frac{d}{dr}(rB(r)) = 1 \tag{2.89}$$

if we integrate (2.89) we get:

$$rB(r) = r + \text{Constant.} \tag{2.90}$$

or

$$B(r) = 1 + \frac{\text{Constant.}}{r} \tag{2.91}$$

To find the constant of integration we recall that at great distances from a central mass M, the component $g_{tt} = -1 - 2\phi = -B(r)$, where $\phi = -MG/r$ is the Newtonian potential. As a result, we have:

$$B(r) = 1 - \frac{2MG}{r} \tag{2.92}$$

and

$$A(r) = \left[1 - \frac{2MG}{r}\right]^{-1}. \tag{2.93}$$

By using (2.92) and (2.93) in (2.80) we get the Schwarzschild solution. In section 3.4 we will find a spherically symmetric solution for equation (2.42) which represents the metric inside the star.

Chapter 3

Numerical Solutions of the Cosmological Equations

So far, in the second chapter, we have found linear and nonlinear ordinary differential equations that represent the cosmological model and the relation between the scale factor and time for different kinds of matter. Also we have found a nonlinear ordinary differential equation that represents the relation between the radius of a giant star and time when this star collapses to form a black hole. In this chapter, we are going to talk about equations that we will solve and then find solutions for these equations. We will display the results that come from solving these equations by directly plotting the explicit solutions and plotting the numerical solutions of the differential equations that we did not find analytical solutions for them.

3.1 The limiting State

Let's begin with equation (2.50), the solution of this equation with the use of Matlab is shown in (appendix A) where $b = \frac{\Lambda}{12}$ which is:

$$a = a_0 \cosh(\sqrt{\frac{\Lambda}{12}}t) \tag{3.1}$$

If k = +1, let's look at equation (2.53). If we introduce a new variable $A = \dot{a}^2 + k$ then $\frac{dA}{da} = 2\dot{a}\frac{d\dot{a}}{da}$. Now, $\ddot{a} = \frac{d\dot{a}}{dt} = \frac{d\dot{a}}{da}\frac{da}{dt} = \frac{d\dot{a}}{da}\dot{a}$. As a result, $\frac{dA}{da} = 2\ddot{a}$. In terms of the new variable, equation (2.53) becomes:

$$\frac{dA}{da} = 2\frac{A}{a} \tag{3.2}$$

if we integrate (3.2) we get the following equation:

$$A = ba^2 \tag{3.3}$$

where b is a constant. By substituting the value of $A = \dot{a}^2 + k$ in equation (3.3) and putting k = +1 we get:

$$\dot{a}^2 = ba^2 - 1 \tag{3.4}$$

which is the same as (2.52) with $b = \frac{\Lambda}{12}$. This equation yields:

$$dt = \frac{da}{\sqrt{ba^2 - 1}} \tag{3.5}$$

By integration, the solution of (3.4) is found to be:

$$a = \frac{\cosh\sqrt{b}t}{\sqrt{b}} \tag{3.6}$$

For simplicity and because we want to know the shape of the solution I will solve equation (3.4) by putting b =1. The solution of this equation is (see appendix A):

$$a = \left(\frac{1}{2} + \frac{1}{2} e^{2t} \right) e^{-t} = \cosh(t) \tag{3.7}$$

this solution is shown in the figure (3.1);

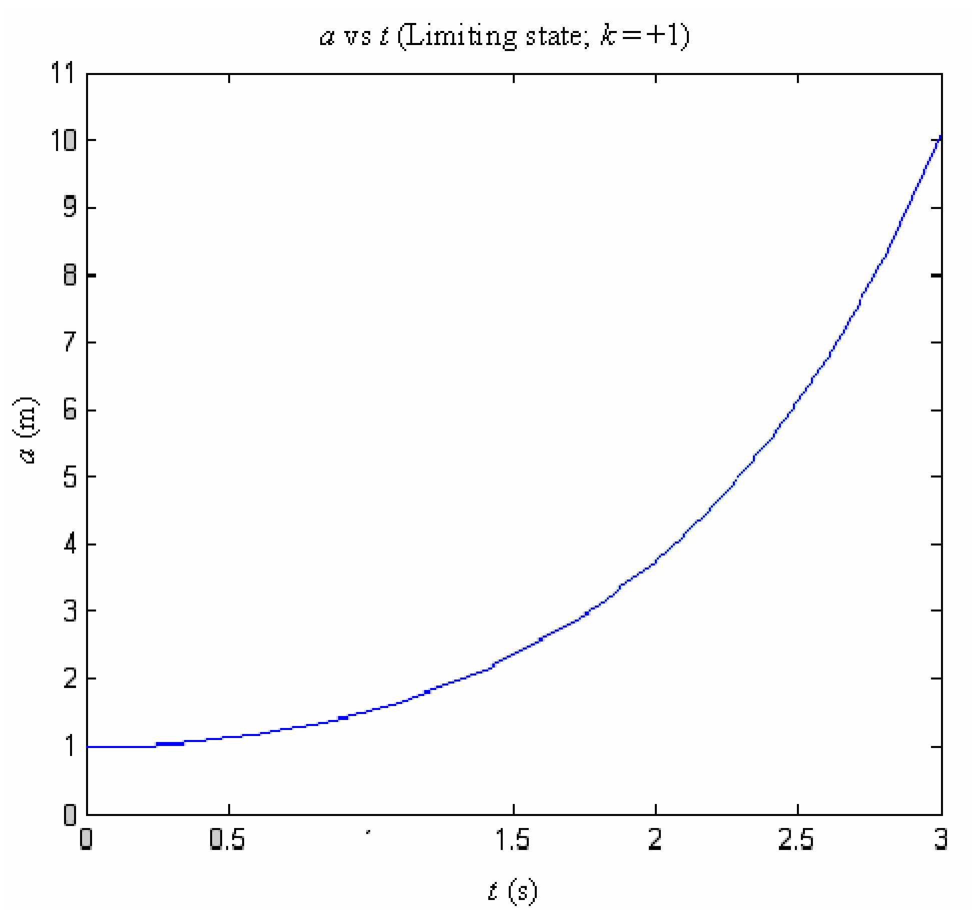

Figure (3.1)   Scale factor versus time (limiting state; k=+1)

To create a 3-D plot of this relation we use cylindrical coordinates and put $a = \sqrt{x^2 + y^2}$. According to that, equation (3.7) becomes: $t = \cosh^{-1}\sqrt{x^2 + y^2}$. Figure (3.2) shows the 3-D plot;

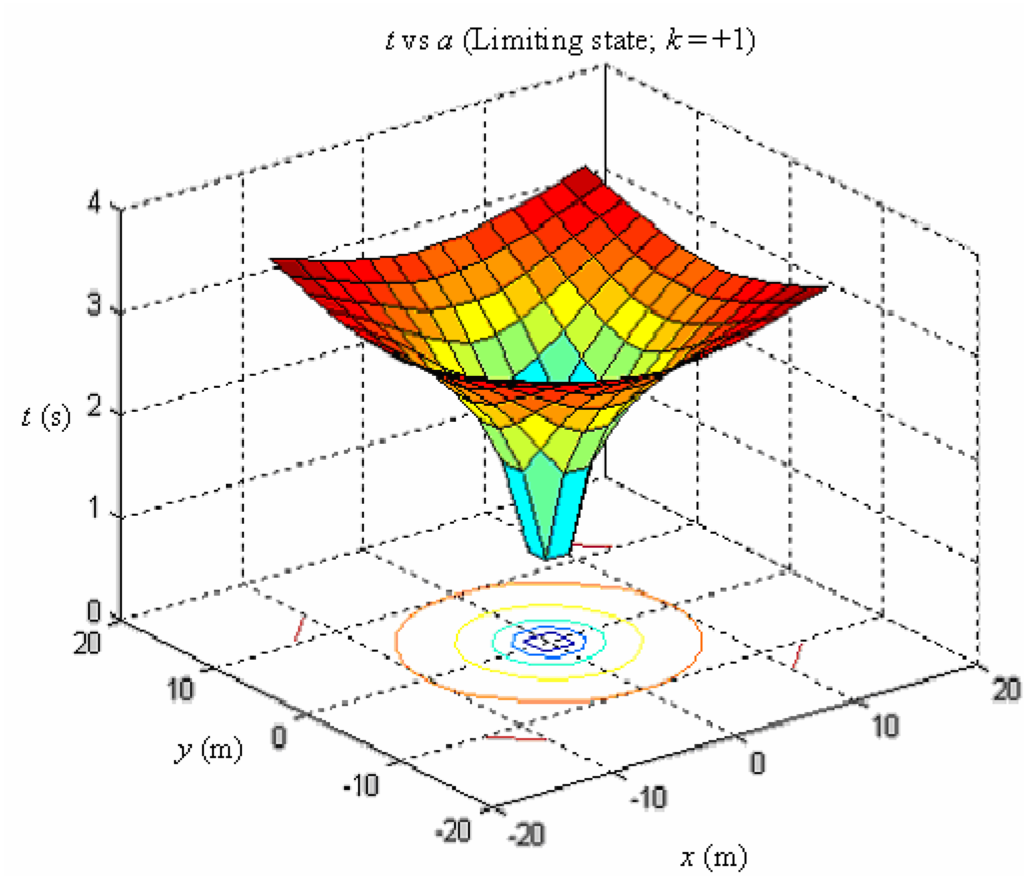

Figure (3.2) 3-D plotting of scale factor versus time (limiting state; k=+1)

For the case $k = 0$, the solution of equation (2.52) where $d = \dfrac{\Lambda}{12}$ and $a(0) = a0$, is given by (see appendix A):

$$a = a_0 e^{\sqrt{d}t} = a_0 e^{\sqrt{\frac{\Lambda}{12}}t}$$

Or

$$t = \sqrt{\frac{12}{\Lambda}} \ln\left(\frac{\sqrt{x^2 + y^2}}{a_0}\right) \qquad (3.8)$$

where $a_0 = \dfrac{1}{\sqrt{\Lambda}}$. To plot the solution, we put $\Lambda = 10^{-8} m^{-2}$. The solution is shown in figure (3.3).

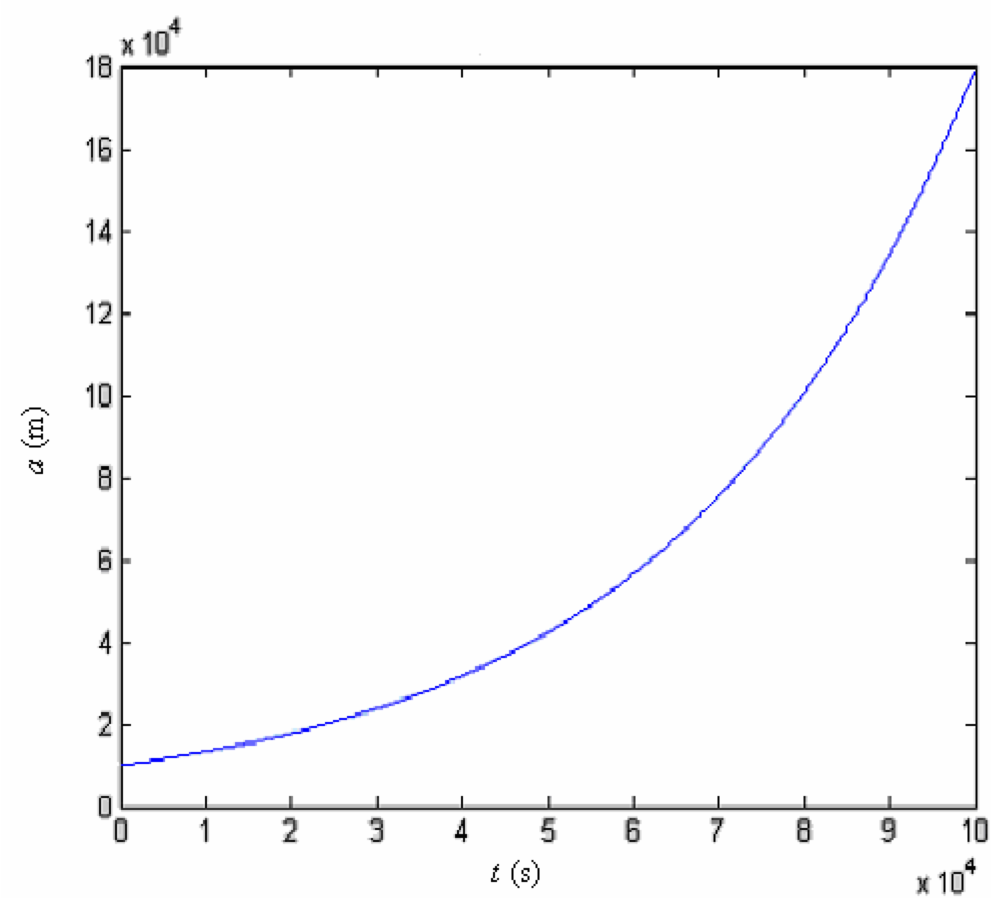

Figure (3.3) Scale factor versus time (limiting state; k=0)

The 3-D plotting of this relation is shown in figure (3.4).

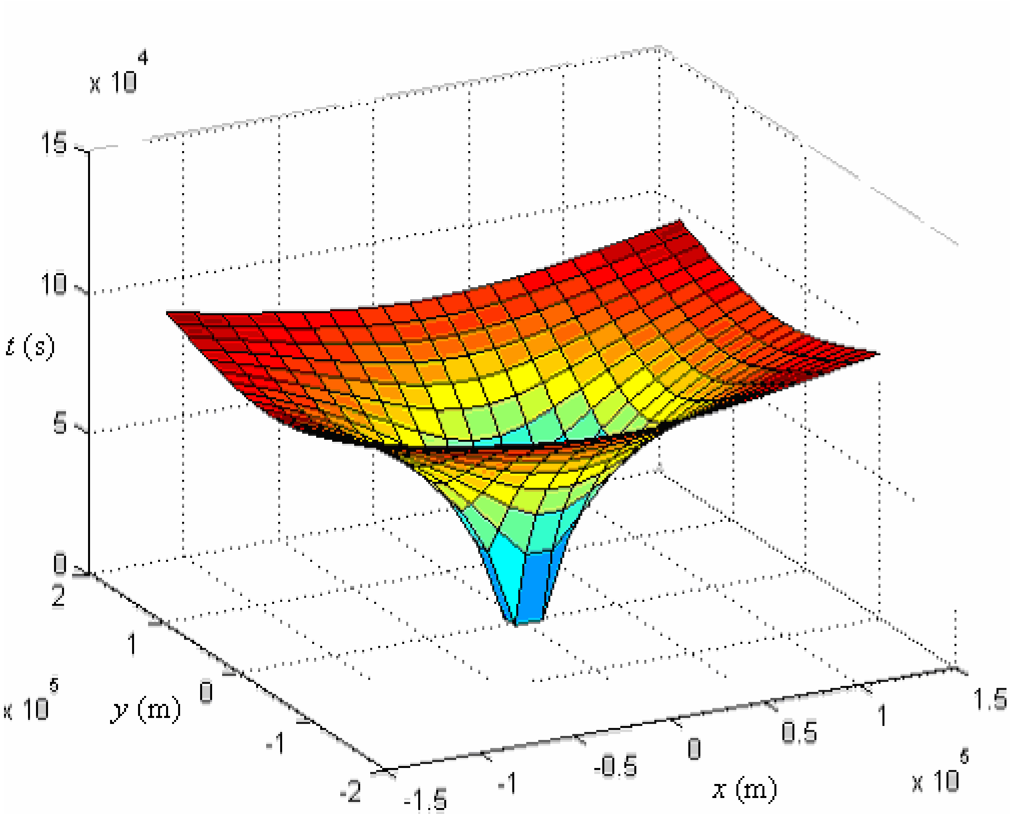

Figure (3.4) 3-D plotting of scale factor versus time (limiting state; k=0)

For the case k = -1, equation (3.5) becomes $dt = \dfrac{da}{\sqrt{ba^2 + 1}}$

the integration of this equation gives:

$$a = \frac{\sinh\sqrt{b}t}{\sqrt{b}} \tag{3.9}$$

This solution is not singular. The singularity of this solution at $t = 0$ is a coordinate singularity not a real singularity. The solution is shown in figure (3.5) for $b = 1$.

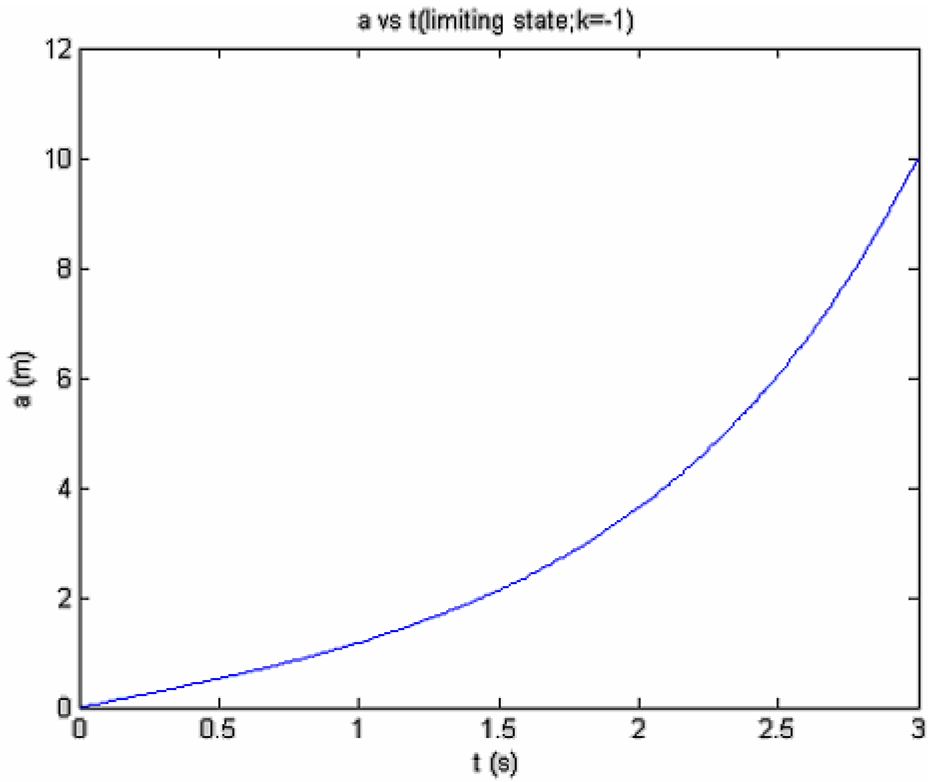

Figure (3.5) Scale factor versus time (limiting state; k=-1)

The 3-D plotting of this relation is shown in figure (3.6).

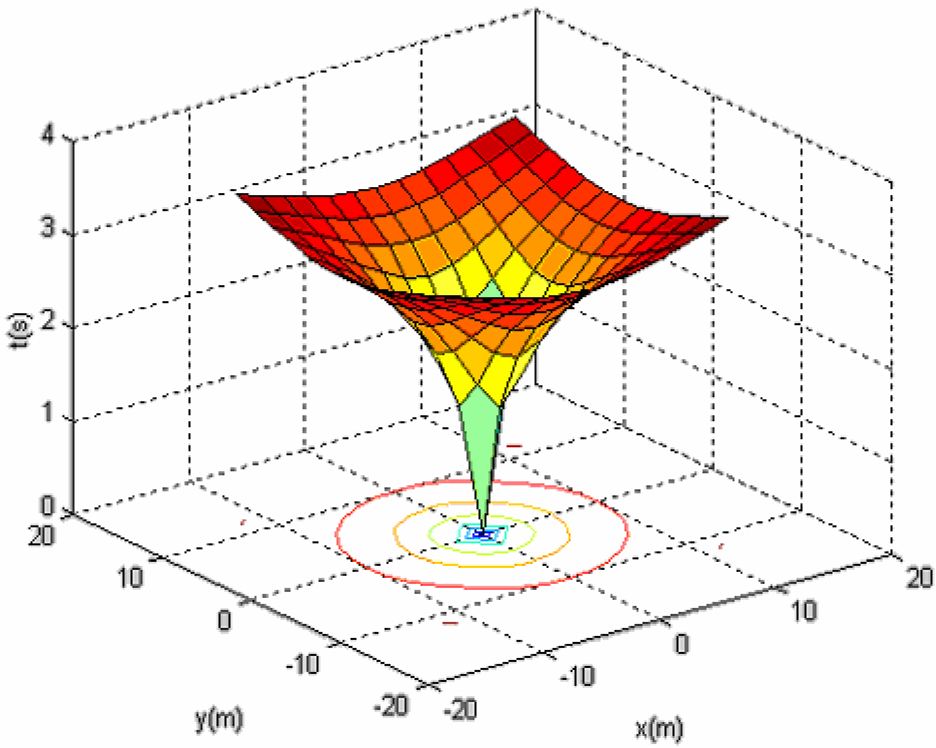

Figure (3.6) 3-D plotting of scale factor versus time (limiting state; k=-1)

3.2 The radiation dominated universe

In the case (k = +1), we will solve equation (2.60). If k = +1, this equation becomes:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + \frac{\Lambda}{12}a^2 - 1 \qquad (3.10)$$

where $\rho = \rho_r = \rho_r^0 \dfrac{a_0^4}{a^4}$. This equation can be written as:

$$\dot{a}^2 = \frac{P}{a^2} + Qa^2 - 1 \tag{3.11}$$

in this equation , $P = \frac{8\pi G \rho_r^0 a_0^4}{3} \approx 1\times 10^{46}$ and $Q = \frac{\Lambda}{12} \approx 1\times 10^{-52}$ where G is the gravitational constant which is equal to $6.67\times 10^{-11} N.m^2/Kg^2$, ρ_r^0 is the radiation density which is approximately equal to $1\times 10^{-31} Kg/m^3$, a_0 is the present value of the scale factor which is approximately equal to $1\times 10^{26} m$, and Λ is the curvature constant. We here used the present value of the curvature constant which is equal to $1\times 10^{-52} m^{-2}$. By setting the rate of expansion $\dot{a} = 0$ in equation (3.11) we have:

$$a_{\pm}^2 = \frac{1 \pm \sqrt{1-4PQ}}{2Q} \tag{3.12}$$

If we want $a_{\pm}$ to be real, PQ must be less than $\frac{1}{4}$. For the above values of P and Q, the value of $a_+ = 1\times 10^{26} m$, and $a_- = 1\times 10^{23} m$. For different values of P and Q the values of a_+ and a_- are shown in the following table:

P(m^4/s^2)	Q(s^{-2})	a_+ (m)	a_- (m)
10^{46}	10^{-52}	10^{26}	10^{23}
1	0.1	2.98	1.06
0.8	0.2	2	1
0.6	0.3	1.6	0.89

The numerical solution of equation (3.11) for those values of P and Q is shown in figure (3.7a), (3.7b), and (3.7c):

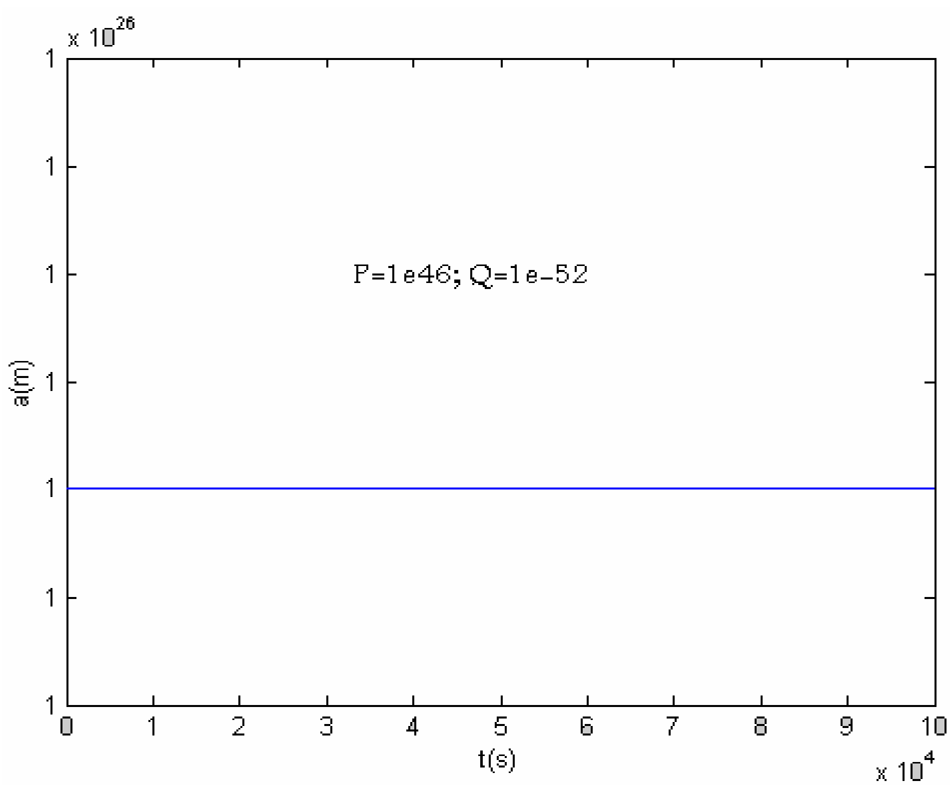

Figure (3.7a) Scale factor versus time (radiation dominated universe, *k* = +1)

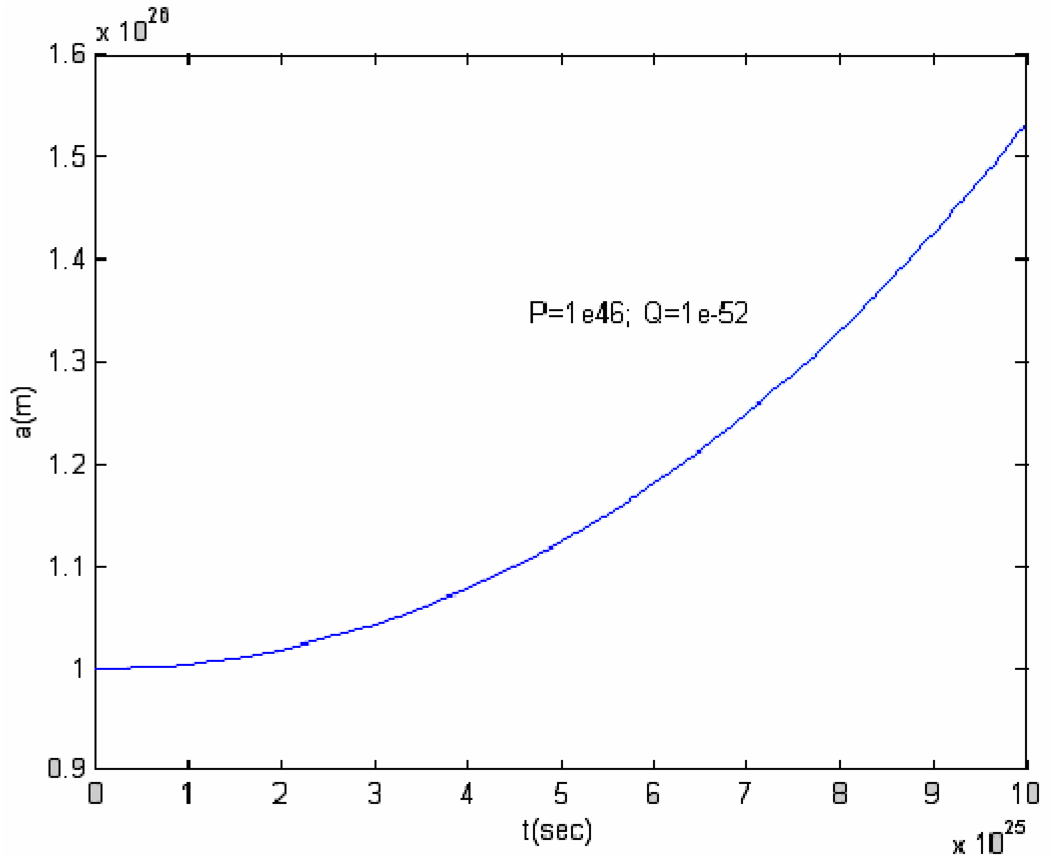

Figure (3.7b) Scale factor versus time (radiation dominated universe, *k* = +1)

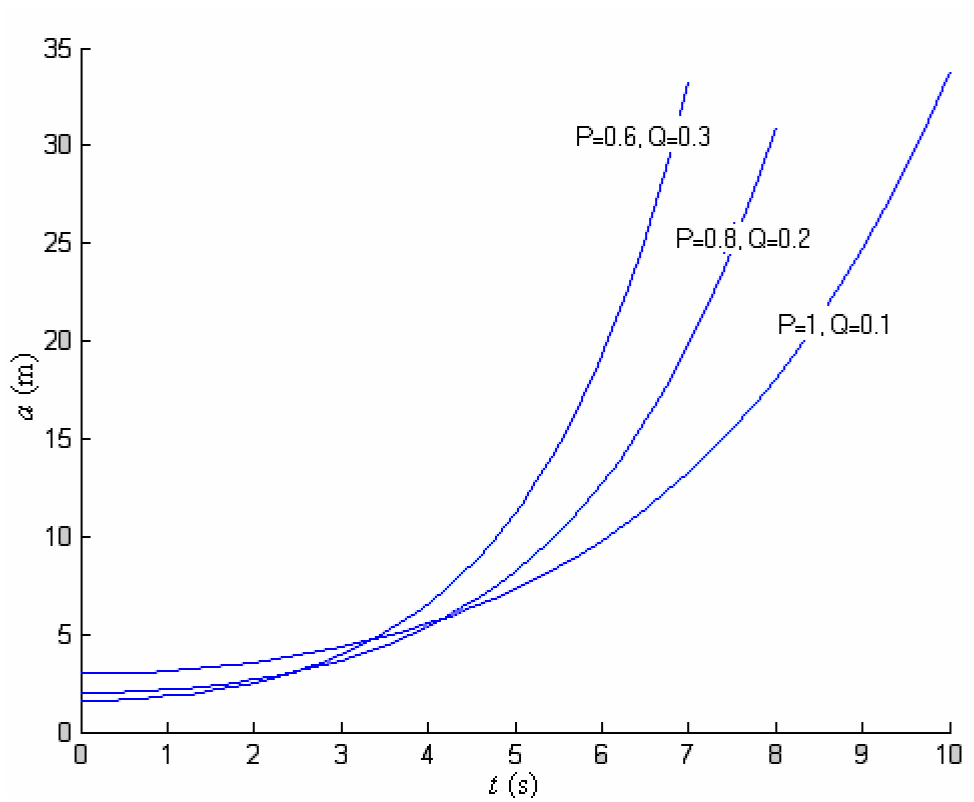

Figure (3.7c) Scale factor versus time (radiation dominated universe, *k* = +1)

the value of the scale factor in figure (3.7a) does not change since the range of time is short but if we take a large scale, the value of the scale factor will change and the graph will be more obvious as in figure (3.7b). The initial condition in this case is a_+ because any value less than a_+ will produce imaginary solutions.

If $k = 0$, equation (2.60) becomes:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + \frac{\Lambda}{12}a^2 \quad \text{Or} \quad \dot{a}^2 = \frac{P}{a^2} + Qa^2 \tag{3.13}$$

where $\rho = \rho_r^0 \dfrac{a_0^4}{a^4}$. The numerical solution of this equation for the above values of ρ_r^0, a_0, and Λ (here $P = \dfrac{8\pi G\rho_r^0 a_0^4}{3} \approx 1\times10^{46}$ and $Q = \dfrac{\Lambda}{12} \approx 1\times10^{-52}$) is shown in figure (3.8).

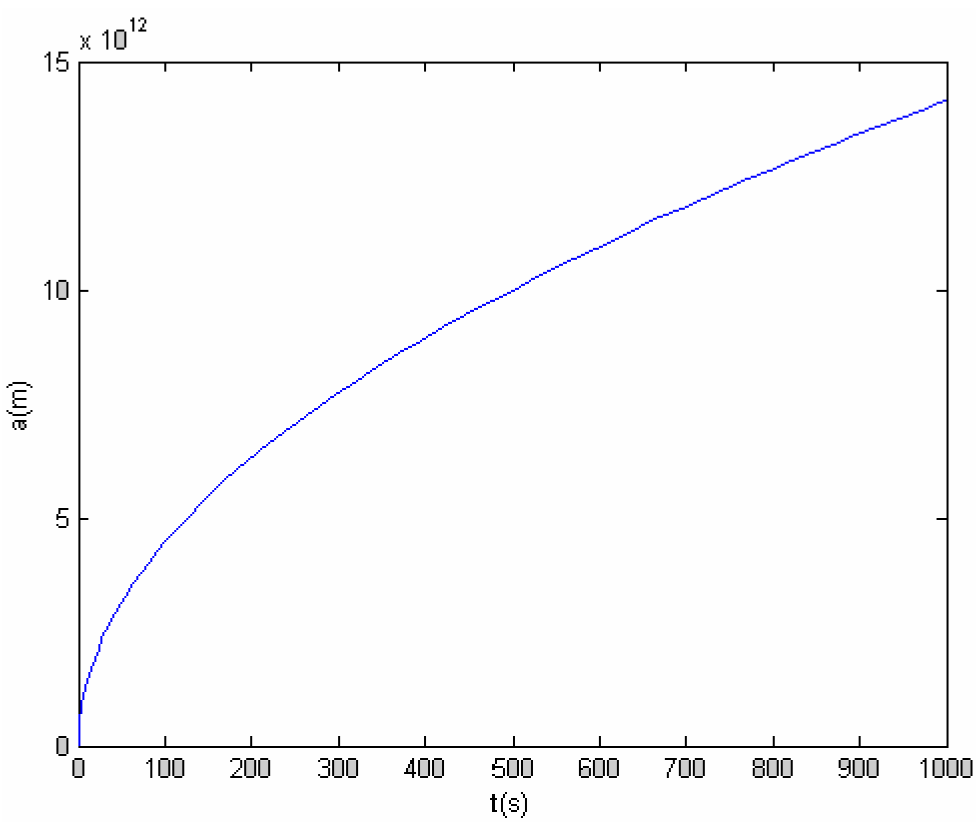

Figure (3.8) Scale factor versus time (radiation dominated universe, *k* = 0)

The square of the scale factor as a function of time is shown in figure (3.9);

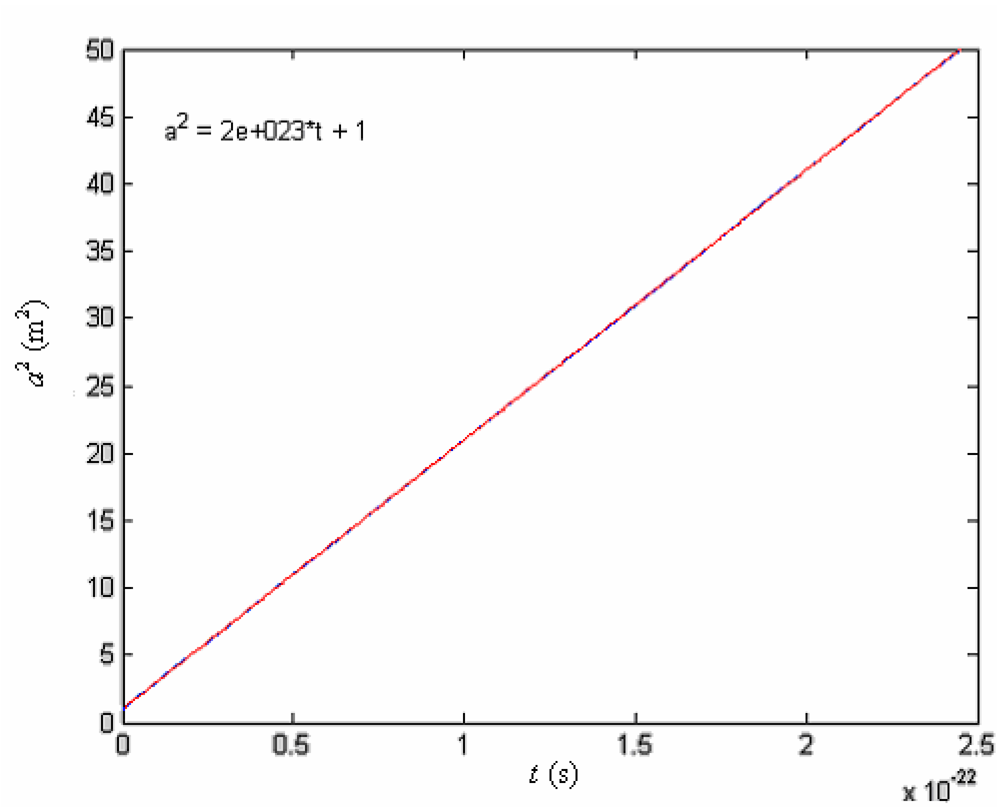

Figure (3.9) Square of the scale factor versus time

(radiation dominated universe, *k* = 0)

The linear fitting of the solution is (see figure (3.9)):

$$a^2 = 2\times10^{23}t + 1 \text{ or } a = (2\times10^{23}t + 1)^{1/2} \qquad (3.14)$$

If k = -1, equation (2.60) becomes:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + \frac{\Lambda}{12}a^2 + 1 \quad \text{Or} \quad \dot{a}^2 = \frac{P}{a^2} + Qa^2 + 1 . \tag{3.15}$$

The numerical solution of equation (3.15) for P=2 and Q=1is (see appendix A):

$$a = \pm\frac{1}{2}\left(-2 + 5.29\sinh(2t + 0.973)\right)^{\frac{1}{2}} \tag{3.16}$$

this solution is shown in figure (3.10).

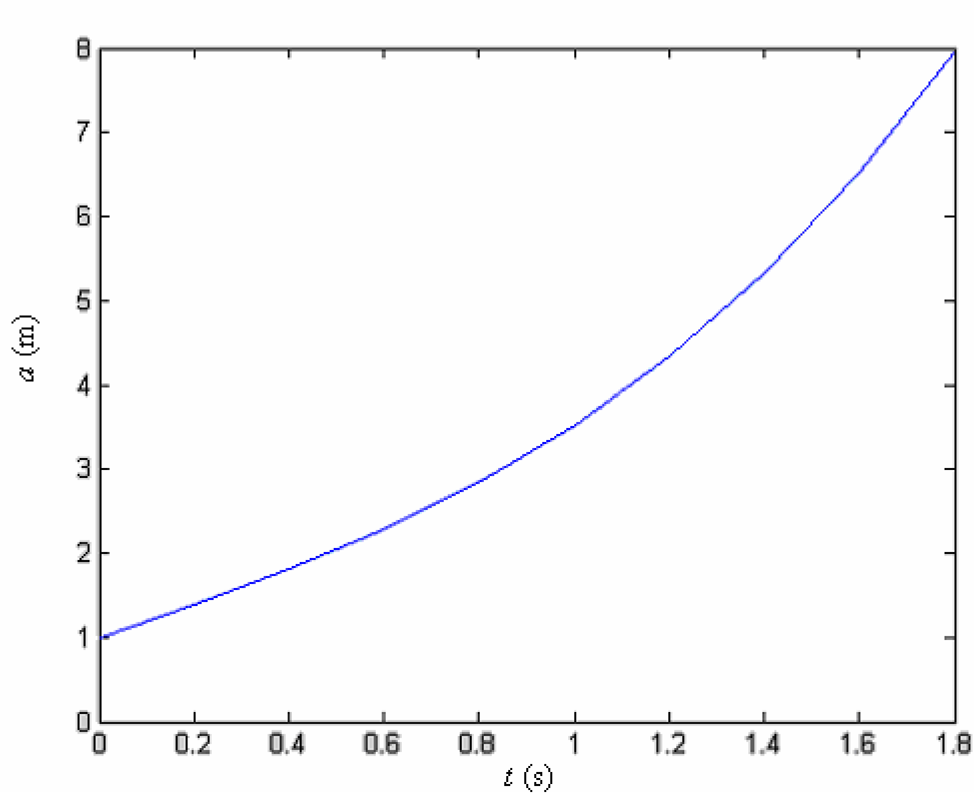

Figure (3.10) Scale factor versus time (radiation dominated universe, *k* = -1)

If we change the values of P and Q, we will obtain the same solution but only with different values of the constants in the solution. We choose any number greater than zero as the initial conditions in (k = 0) and (k = -1) cases to avoid dividing by zero which produces unknown solutions. In figure (3.9) we choose a short time interval in order to display the initial condition clearly on the graph and to show that the solution is not singular.

3.3 The matter dominated universe

In this case, we will continue from equation (2.75). To avoid dealing with quantities that are depending on time (unknown functions of time such as *f(t)* in equation (2.66) and γ in equations (2.70) and (2.71)) we will go in the following procedure:

It is possible to expand Y in powers of $\frac{1}{\Lambda}$:

$$Y = Y_0 + \frac{Y_1}{\Lambda} + \frac{Y_2}{\Lambda^2} + \ldots \tag{3.17}$$

then Y_0 satisfies the following equation:

$$\frac{dY_0}{da} + \frac{1}{a} Y_0 = 0 \tag{3.18}$$

which has the following solution:

$$Y_0 = \frac{C}{a}. \tag{3.19}$$

At the first order in $\frac{1}{\Lambda}$, Y_1 satisfies the equation:

$$\frac{dY_1}{da}+\frac{1}{a}Y_1-\frac{9C^2}{8a^5}=0 \tag{3.20}$$

which has the following solution:

$$Y_1=\frac{1}{a}\left(D-\frac{3C^2}{8a^3}\right) \tag{3.21}$$

From equations (3.17), (3.19), and (3.21) and by putting $Y=\dot{a}^2+k$ we can write a first order differential equation for the scale factor:

$$\dot{a}^2+\frac{3C^2}{8a^4}-\frac{C+D}{a}=-k \tag{3.22}$$

or

$$\dot{a}^2+\frac{A}{a^4}-\frac{B}{a}=-k \tag{3.23}$$

where *A* and *B* are constants. The sign of *A* is positive since $A=\frac{3C^2}{8}$.

The sign of *B* may be either positive or negative but if we put it negative $\dot{a}$ will be imaginary for k = +1. So in our work we use positive values for *A* and *B*. By setting the rate of expansion $\dot{a}=0$ and k = +1, and substituting numerical values for A and B then equation (3.23) will have two real roots which represent the minimum radius from which the universe starts to expand and the maximum radius at which the universe will stop expanding and collapse. The following table shows the values of the minimum and the maximum radii at different values of *A* an *B*;

A	*B*	Minimum radius(m)	Maximum radius(m)
4	6	0.9236	5.9813
10	5	1.4066	4.9158
9	4	1.5411	3.8412

The solution is shown in figure (3.11a) for different values of *A* and *B*;

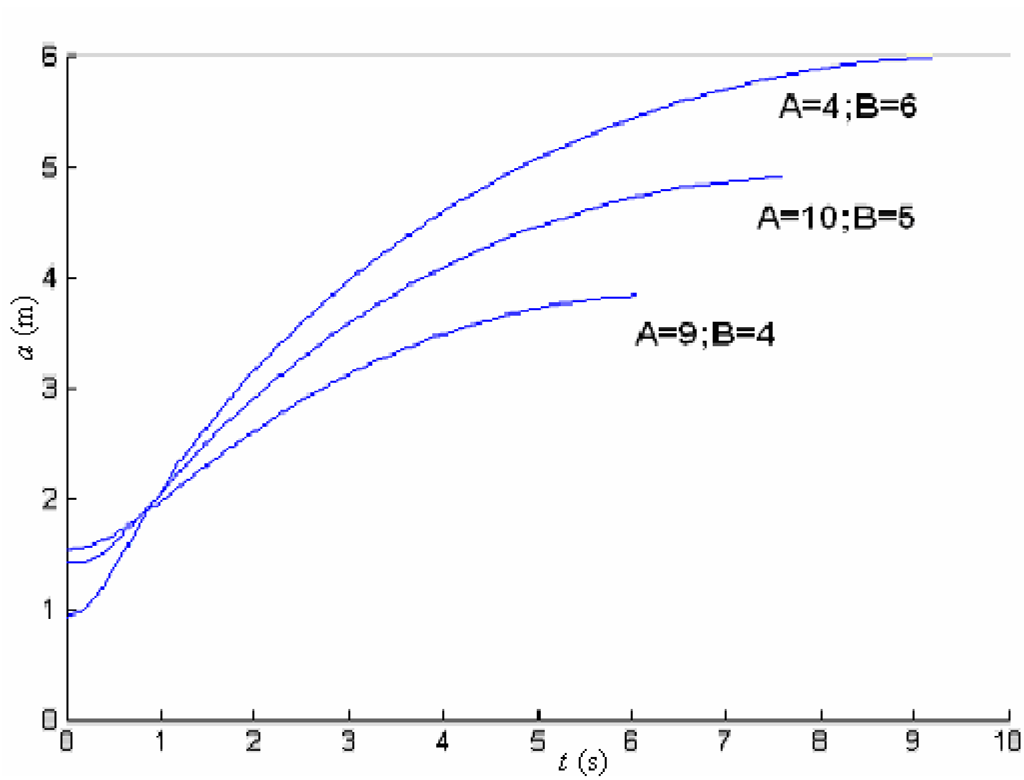

Figure (3.11a) Scale factor versus time (matter dominated universe, k = +1)

For large values of the scale factor and for $k = +1$, equation (3.23) is reduced to $\dot{a}^2 = \frac{B}{a} - 1$. This differential equation represents a Cycloid [30]. This equation comes from the following parametric equations: $t = \frac{B}{2}(t' - \sin t')$ and $a = \frac{B}{2}(1 - \cos t')$ [31]. Figure (3.11b) shows the graph of the scale factor with time according to the parametric equations and for $B = 2$.

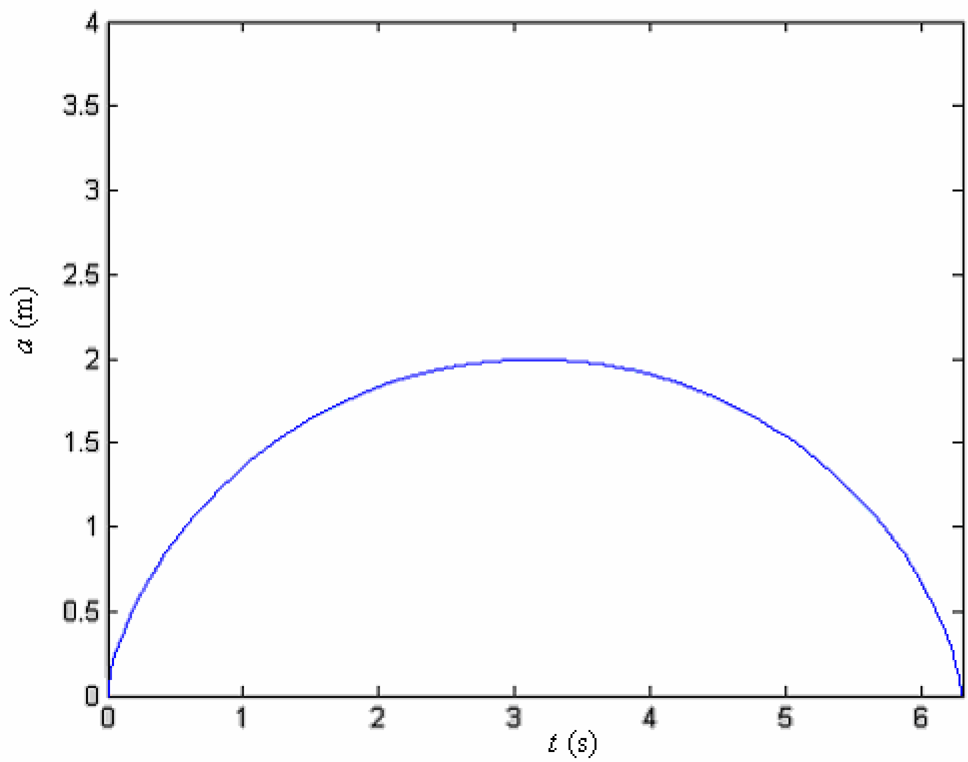

Figure (3.11b) scale factor vs time in the matter dominated universe for k = +1 as predicted by CGR.

The matter density is a function of the scale factor according to equation (2.68); as a result, it is a function of time too. The contraction of equation (2.63) implies that:

$R + 4f(t)\Lambda = 8\pi G\rho$ or $\beta\Lambda - \Lambda(2\beta + 1 - \sqrt{1-\beta^2}) = 8\pi G\rho$.

So since ρ is a function of time then β will be a function of time too. Figures (3.12) and (3.13) show ρ and β as functions of time for *A*=10 and *B*=5;

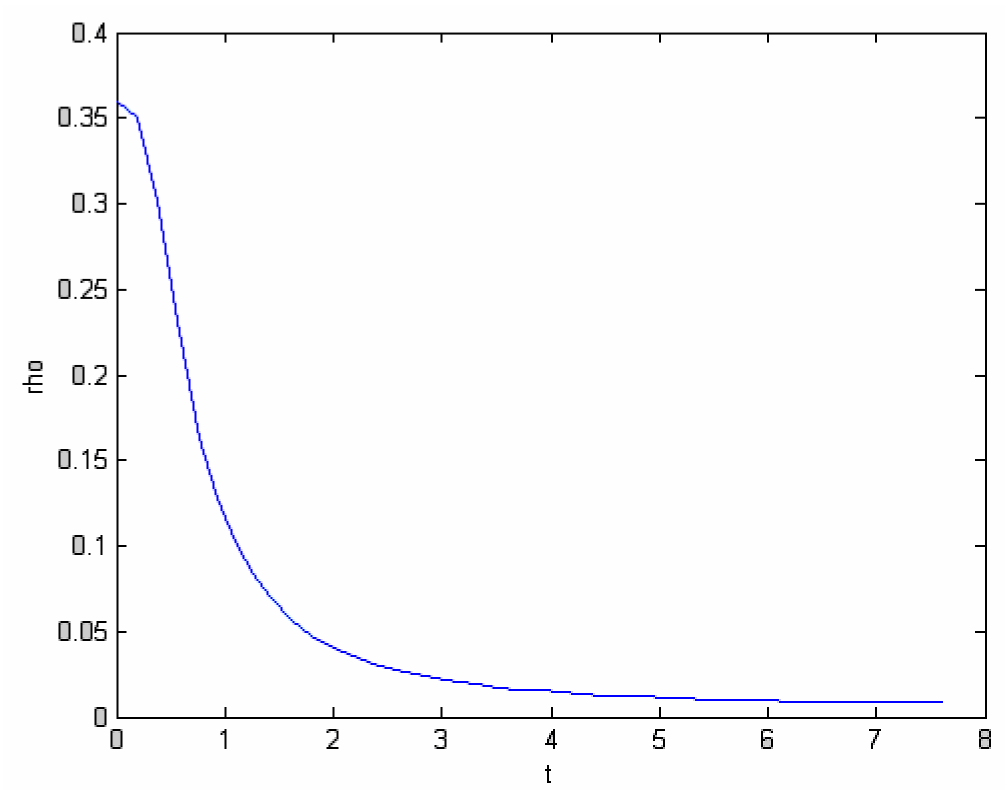

Figure (3.12) matter density versus time (matter dominated universe, *k* = +1)

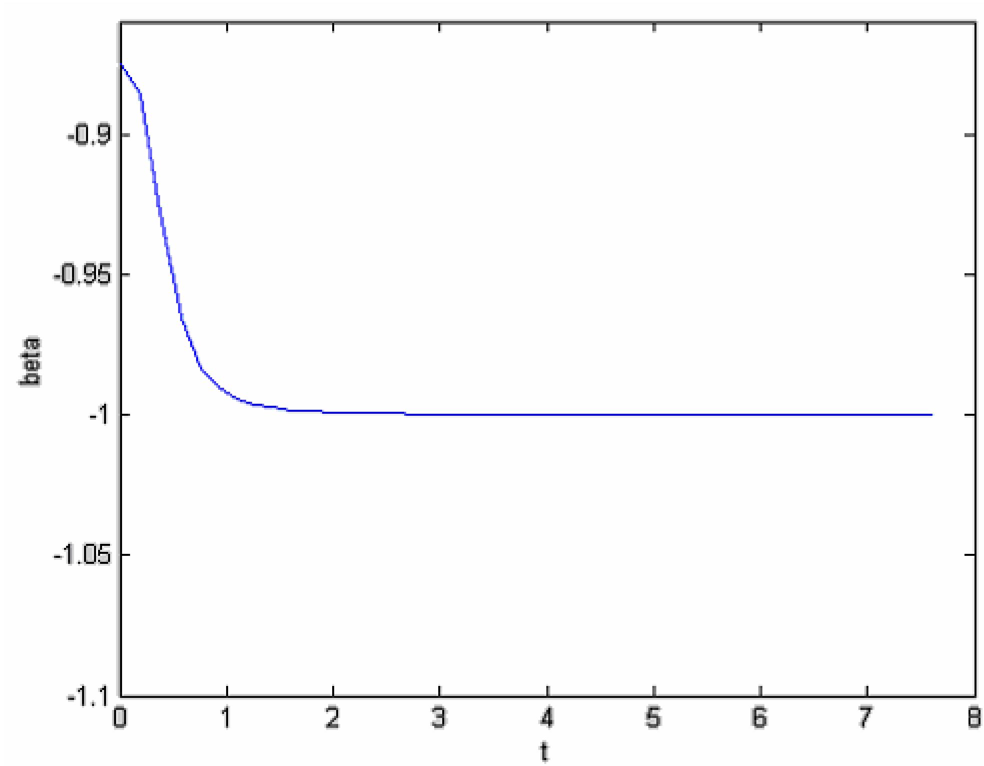

Figure (3.13) Beta versus time (matter dominated universe; = +1)

For k=0 and $\dot{a} = 0$, equation (3.23) has one real root which represents the minimum radius from which the universe starts expanding forever. The signs of A and B are also positive for the same reasons discussed above (in the case k = +1). The following table shows the value of the minimum radius at different values of A an B;

A	*B*	Minimum radius(m)
4	6	0.8736
10	5	1.2599
9	4	1.3104

The numerical solution of equation (3.23) for different values of A and B is shown in figure (3.14);

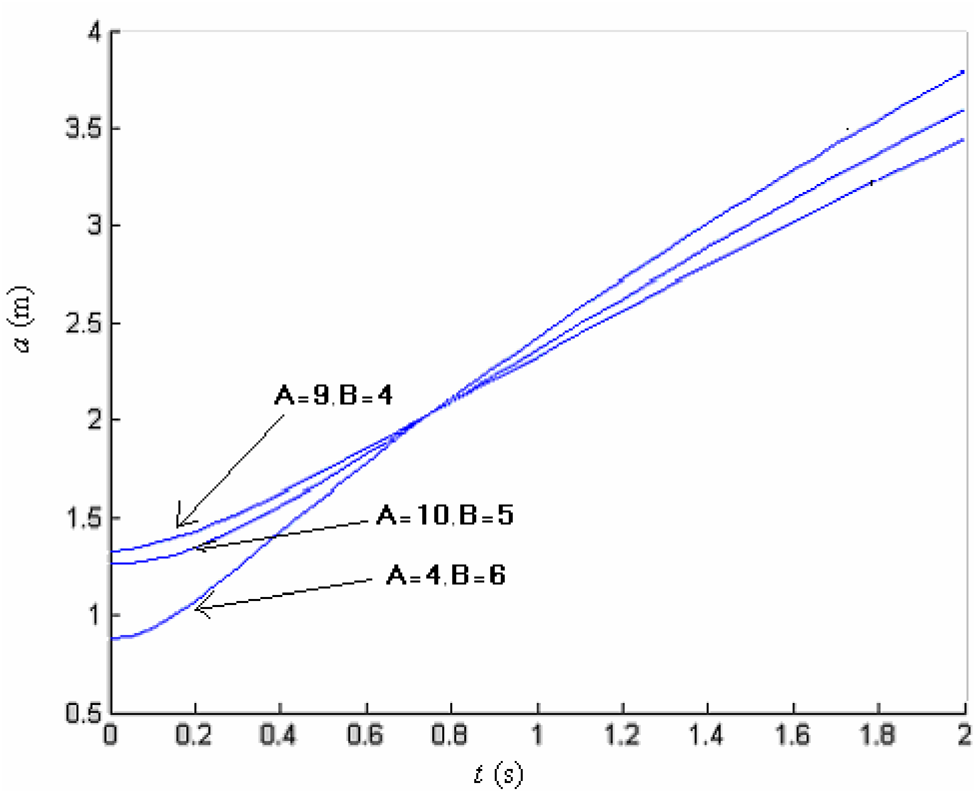

Figure (3.14) Scale factor versus time (matter dominated universe, *k* = 0)

For large values of the scale factor, the numerical solution of equation (3.23) for A=10 and B=5 is shown in figure (3.15);

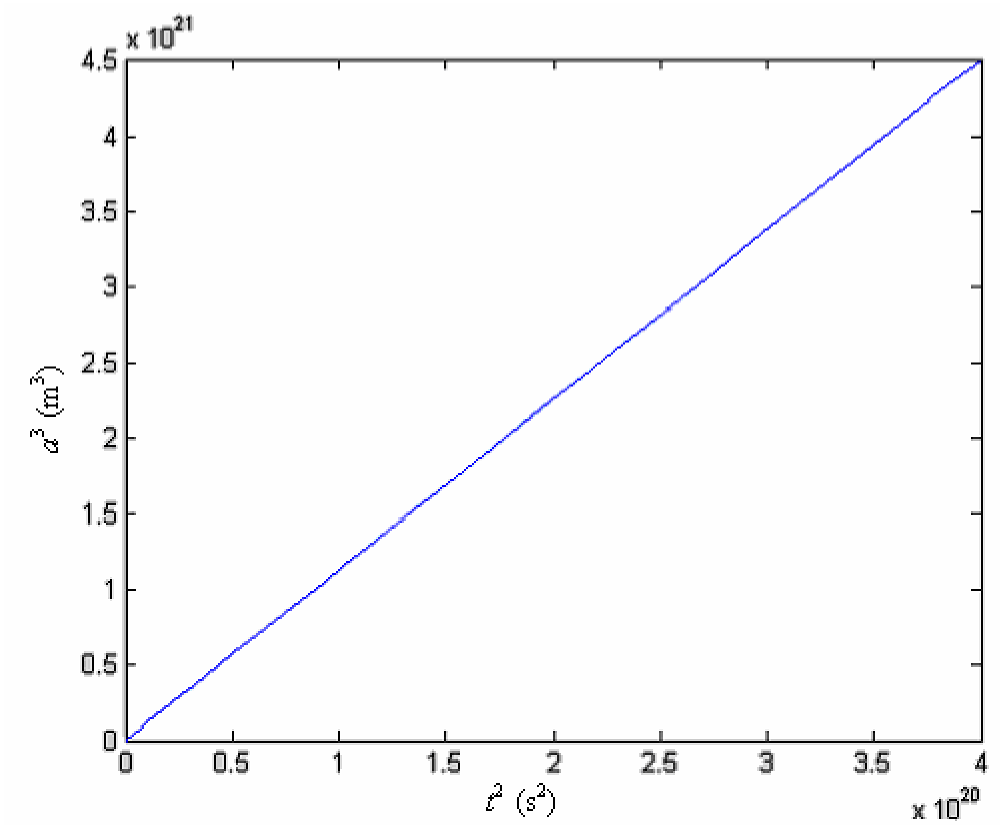

Figure (3.15) The cube of the scale factor versus the square of time

(matter dominated universe, k = 0)

It is obvious from figure (3.15) that:

$$a^3 \approx t^2 \quad (3.24)$$

or

$$a = t^{2/3} \quad (3.25)$$

For k= -1 and $\dot{a} = 0$, the sign of A is positive but the sign of B is either positive or negative. In both cases, equation (3.23) has two real roots, one is positive and the other is negative. The negative real root is ignored since the scale factor is always positive. The positive real root represents the minimum radius from which the universe starts expanding forever. The following table shows the value of the minimum radius at different values of A an B;

A	B	Minimum radius(m)
4	-6	6.0183
10	5	1.1744
9	-4	4.1279

The solution is shown in figure (3.16) for different values of A and B;

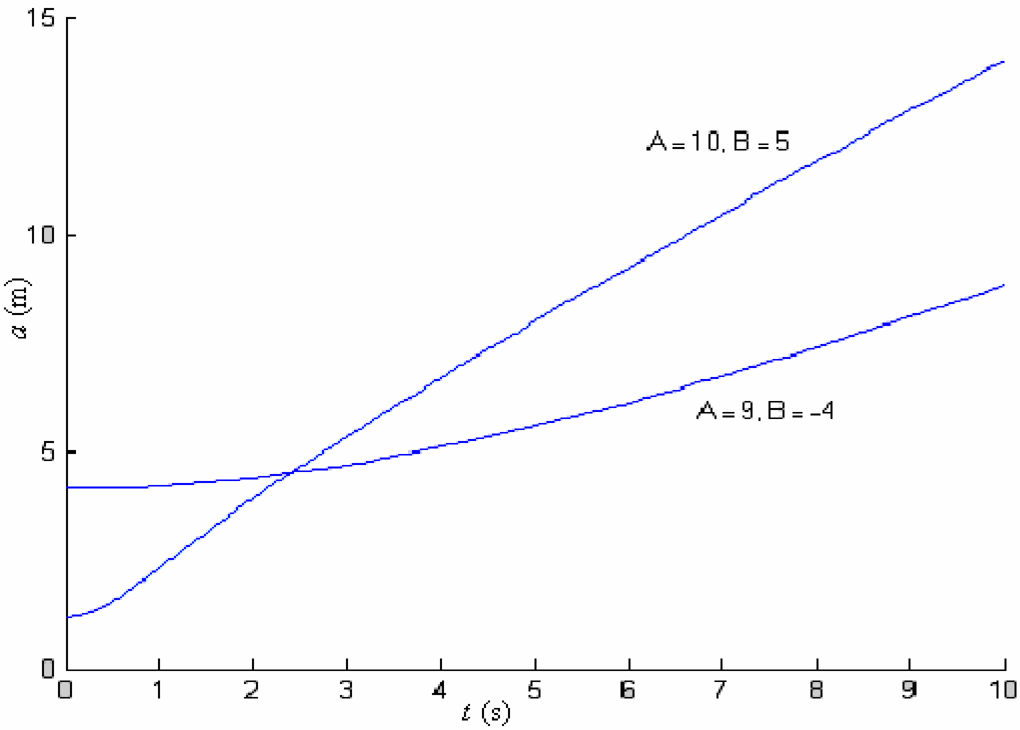

Figure (3.16) Scale factor versus time (matter dominated universe, k = -1)

The three cases are shown together in figure (3.17).

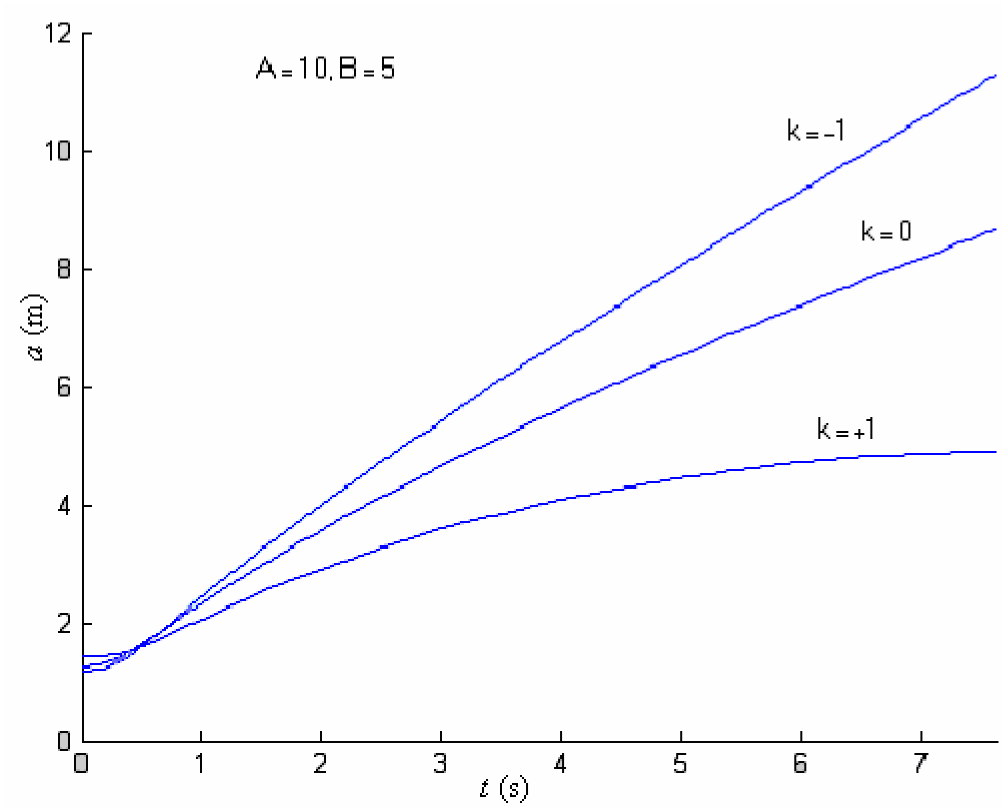

Figure (3.17) Scale factor versus time (matter dominated universe, the three cases)

3.4 The Spherically symmetric solution

The solution of the field equations outside a spherically symmetric massive object will be exactly the Schwarzschild solution (2.11) where the quantity (*2GM*) in this equation is the gravitational radius of the star with mass *M*. Now let's go back to figure (2.3). This figure shows the shape of the solution outside a massive object which is, in this case, the earth. The two-dimensional Schwarzschild solution is:

$$ds^2 = \frac{dr^2}{1-\frac{r_g}{r}} + r^2 d\theta^2 \tag{3.26}$$

where $r_g = 2GM$. If $r \to \infty$ the solution becomes $ds^2 = dr^2 + r^2 d\theta^2$ which is the two-dimensional flat space metric in spherical coordinates. This means that at a very large distance from the massive object the space is flat. In rectangular coordinates, $ds^2 = dx^2 + dy^2 + dz^2$ or $ds^2 = dr^2 + dz^2$. Now let $dz = z'dr$ then equation (3.26) becomes:

$$dr^2 + dz^2 = \frac{dr^2}{1-\frac{r_g}{r}} = \left(1 + z'^2\right)dr^2 \tag{3.27}$$

and

$$z'^2 = \frac{1}{1-\frac{r_g}{r}} - 1 \tag{3.28}$$

this implies that:

$$z' = \sqrt{\frac{r_g}{r - r_g}} \tag{3.29}$$

and since $z = \int z'dr$;

$$z = 2\sqrt{r_g(r - r_g)} \tag{3.30}$$

this solution is shown in figure (3.18).

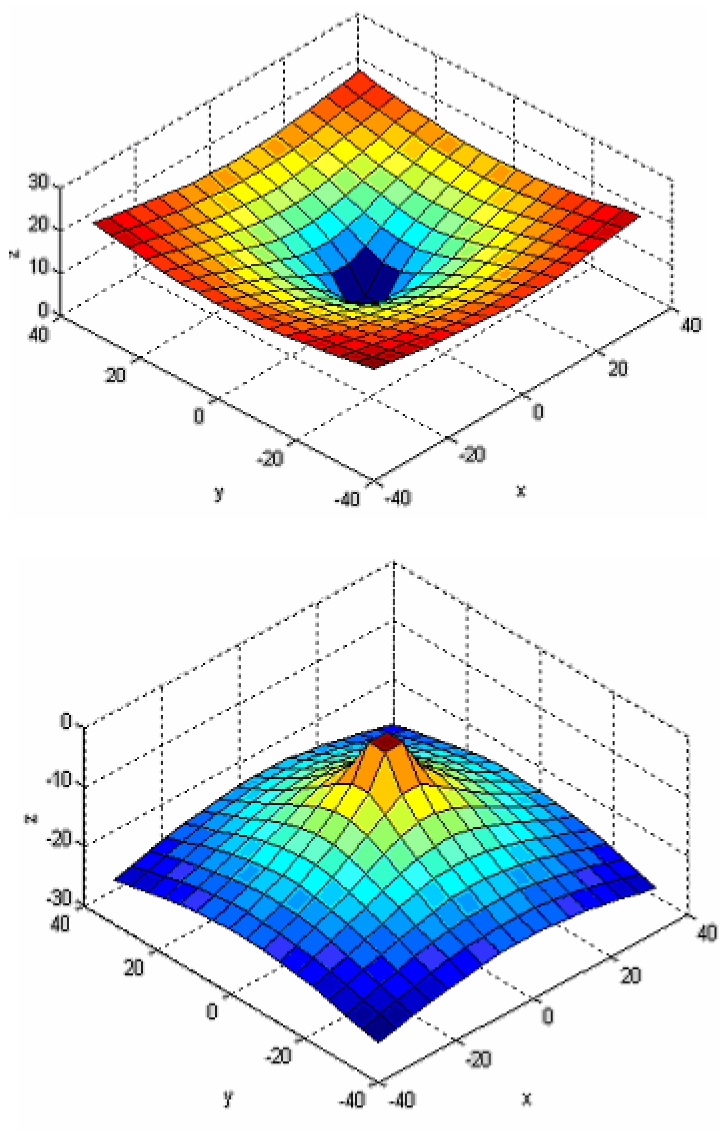

Figure (3.18) Schwarzschild solution

The internal solution (inside the star) is assumed to be homogeneous and isotropic and that the particles inside the star become extremely relativistic after being compressed to densities above the nuclear density. This means that the equation of state becomes that of pure radiation. Now, let's discuss the analogy between the radiation dominated universe with positive curvature and the star. After being compressed to densities above the nuclear density, the particles inside the star become extremely relativistic, so we can consider the star as radiation dominated universe with positive curvature (k = +1) since the star is positively curved because it is a spherical object. We assume that the density and the pressure are functions of time only.

Now, the cosmological equations that govern the radiation dominated universe will surely be suitable for describing the collapsing of the star. So by putting R, which is the radius of the star, instead of a, which is the radius of the universe, in equation (2.60) we get:

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 + \frac{\Lambda}{12}R^2 - 1 \tag{3.31}$$

the density of the collapsing star depends on the radius of the star according to the following equation:

$$\rho = \rho_0 \frac{R_0^4}{R^4} \tag{3.32}$$

where ρ_0, R_0 are the initial density and radius of the collapsing star respectively (the initial density and radius of the remaining mass of the star).

The solution of the differential equation (3.31) is shown in figure (3.7). It is very important to notice that the collapsing star will stop at a minimum nonsingular radius and will never reach a singularity, which is consistent with Penrose diagrams. According to that, no singularity is developed at the center of the collapsing star.

Chapter 4

Analysis and Comments

In this chapter, we will analyze the results that we got in chapter 3 and we will check if they agree with LCH. First we will discuss the limiting state case. In this case LCH suggests that when all curvature invariants approach their limiting value, a nonsingular de Sitter solution is taken on. So, at the limiting state we will have a de Sitter space. As we can see, the solutions (3.1) and (3.6) are the same as equation (2.3). From the plots of the limiting state case, the solutions are not singular, the universe begins from a nonsingular state (the radius of the universe is not zero) at $t = 0$ and expands. On the contrary, CGR predicts that the universe started from a singular state and expanded. In the case ($k = -1$) the solution is not singular. The singularity of this solution at $t = 0$ is not a real singularity. The most important case is when k = +1 which tells us that the universe will recollapse after it reaches the maximum radius and when it collapse to the minimum radius, we will reach a de sitter solution since the curvature approaches its limiting value. The de sitter solution in this case is not singular so the universe will reach a minimum radius and expand again.

In the radiation dominated universe when k = +1, the universe has a closed topology. It bounces from a nonsingular state at a minimum radius and expands. It will never reach the big crunch singularity as suggested by CGR. In this case the behavior of the universe agrees with Penrose diagrams and with LCH which suggests that, for spherically symmetric spacetime, a collapsing universe will not end up in a big crunch, but it will bounce and expand. The reason is that when the radius of the universe becomes small the density becomes high and so is pressure because the matter in the universe will become relativistic, then the outward push of pressure will overcome the inward pull of gravity, and this will cause the expansion of the universe. When $k = 0$, the solution that we found, (equation (3.14)), is the nonsingular version of equation (2.62) which is suggested by CGR. Since the universe has open topology, it will start from a nonsingular state and expand forever. When $k = -1$, the behavior of the universe is the same as the case when $k = 0$ because, in this case, it has an open topology too.

In the matter dominated universe, and when $k = +1$, the universe expands from a nonsingular radius and reaches a maximum radius and then stop and collapse under the influence of gravity. The behavior of the universe can be explained as follows: as the universe expands the pressure drops down, as a result the force of gravity between the matter components of the universe including the dark matter will overcome the force from pressure and this will cause the universe to stop and collapse again. The universe will go on collapsing until the radius of the universe become small. The universe at that moment will become a radiation dominated universe and it will bounce at ($\dot{a} = 0; a = a_+$) and expand again. This behavior of the universe in this case is consistent with Penrose diagrams.

At large values of the scale factor, equation (3.23) will be reduced to Einstein's equation ($\dot{a}^2 = \frac{B}{a} - k$). The solution of this equation when k = +1 is shown in figure (2.6). The universe will stop and collapse (see [17], p481-483). When k=0, and for large values of the scale factor, the solution that we found, (equation (3.25)), is the same solution which is suggested by CGR [17] which is ($a = t^{2/3}$). Since the universe has open topology, it will start from a nonsingular state and expand forever. When k = -1, the behavior of the universe is the same as the case when k = 0 because, in this case, it has an open topology too. When we compare between figures (2.6) and (3.17) we find that the two figures agree with each others in the region where the scale factor is large and the curvature is small since equation (2.42) is reduced to EFEs at low curvatures. But at high curvatures and at small values of the scale factor, the difference between the two figures becomes clear. The solutions in figure (2.6) are singular but the solutions in figure (3.17) are not. The idea is: the solutions of equation (2.42) are the same as Einstein's solution at low curvatures but they are different at high curvatures.

For spherically symmetric solutions, there will be no singularity inside the event horizon; instead, a de Sitter universe will be reached as we see in Penrose diagrams. The giant collapsing star will collapse to a minimum radius and expand again. The reason is similar as that in the radiation dominated case with closed topology, that when the radius of the collapsing star becomes small the density becomes high and the particles inside the star will be relativistic and the pressure becomes high. In that case the pressure inside the star will overcome the force of gravity, and this will cause the expansion of the star. CGR predicts a singularity in the center of the black hole, but LCH suggests that the radius of the star will reach a minimum nonsingular value and will never reach zero resulting in a nonsingular black hole. Finally, the plot in figure (3.18) which represents the solution (3.30) is consistent with figure (2.3), the curvature decreases as we go away from the black hole or any other massive object and increases near the object. At last, we conclude that all solutions are nonsingular and this is consistent with our hypothesis “the Limiting Curvature Hypothesis”.

Appendix A

Numerical Solutions

1. Limiting State

```
// solving equation (2.50)

>> a=dsolve('D2a-b*a=0','a(0)=a0','Da(0)=0')

a =

a0*cosh(b^(1/2)*t)
```

$k = +1$

```
>> a=dsolve('Da=(a^2-1)^(1/2)','a(0)=1')

a =

(1/2+1/2*exp(2*t))*exp(-t)

// 2-D plotting

>> t=0:0.1:3;

>> a=cosh(t);

>> plot(t,a)

// 3-D plotting

>> [x,y]=meshgrid(-13:2:13);

>> z=acosh(sqrt(x.^2+y.^2));

>> surfc(x,y,z);
```

$k = 0$

```
>> a=dsolve('Da=(d*a^2)^(1/2)','a(0)=a0')

a =

exp(d^(1/2)*(t+log(a0)/d^(1/2)))

// 2-D plotting

>> t=0:100:100000;

>> a=10000*exp(2.88688e-5*t);

>> plot(t,a)

// 3-D plotting

>> [x,y]=meshgrid(-134349:1.4142e4:134349);

>> z=x.^2+y.^2;

>> a=sqrt(z);

>> b=a/10000;

>> c=log(b);

>> d=c*3.4639e4;

>> surf(x,y,d);
```

$k = -1$

```
>> t=0:0.01:3;

>> a=sinh(t);

>> plot(t,a)
```

```
// 3-D plotting
>> [x,y]=meshgrid(-12:2:12);
>> z=asinh(sqrt(x.^2+y.^2));
>> surfc(x,y,z);
```

2. Radiation Dominated Universe

$k = +1$

```
// finding a + and a - for different values of P and Q
solve('1+0.1*a^4-a^2=0','a')
ans =
[ -1.0616104058422671258159534942517]
[  1.0616104058422671258159534942517]
[ -2.9787553350699041400414946820376]
[  2.9787553350699041400414946820376]

>> solve('0.8+0.2*a^4-a^2=0','a')
ans =
[  1.]
[  2.]
[ -2.]
[ -1.]
```

```
>> solve('0.6+0.3*a^4-a^2=0','a')

ans =

[ -.88586091627211424411238045875781]

[  .88586091627211424411238045875781]

[ -1.5964284419775486855698043212680]

[  1.5964284419775486855698043212680]

****************************************

function adot=rad_dom_1(t,a)

adot=(1/a^2+0.1*a^2-1)^(1/2)

****************************************

function adot=rad_dom_11(t,a)

adot=(0.8/a^2+0.2*a^2-1)^(1/2)

****************************************

function adot=rad_dom_111(t,a)

adot=(0.6/a^2+0.3*a^2-1)^(1/2)

****************************************

>>[t,a] = ode45('rad_dom_1',[0 10],2.98)

>>[t1,a1] = ode45('rad_dom_11',[0 8],2.0001)

>>[t2,a2] = ode45('rad_dom_111',[0 7],1.6)

>>hold on

>>plot(t,a)

>>plot(t1,a1)
```

```
>>plot(t2,a2)
>>hold off
```

$k = 0$

```
function adot=rad_dom_0(t,a)
adot=(1e46/a^2+1e-52*a^2)^(1/2);
*****************************************************
[t,a] = ode45('rad_dom_0',[0 1e3],1)
plot(t,a)
*****************************************************
for i=1:321
b(i,1)=a(i,1)^2
end
plot(t,b)
```

$k = -1$

```
a=dsolve('Da=(2/a^2+a^2+1)^(1/2)','a(0)=1')
a =
     1/2*(-2-2*7^(1/2)*sinh(2*t-asinh(3/7*7^(1/2))))^(1/2),
    -1/2*(-2-2*7^(1/2)*sinh(2*t-asinh(3/7*7^(1/2))))^(1/2)
```

3. matter dominated universe

$k = +1$

```
//finding the roots of equation (3.23) for A=4; B=6;

>> solve('6*a^3-4-a^4=0','a')

ans =

    5.9813

ans =

    0.9236

ans =

  -0.4525 + 0.7206i

ans =

  -0.4525 - 0.7206i

//finding the roots of equation (3.23) for A=10; B=5;

>>solve('5*a^3-10-a^4=0','a')

ans =

    4.9158

ans =

    1.4066

ans =

  -0.6612 + 1.0045i

ans =

  -0.6612 - 1.0045i

//finding the roots of equation (3.23) for A=9; B=4;

solve('4*a^3-9-a^4=0','a')
```

```
ans =

   3.8412

ans =

   1.5411

ans =

  -0.6912 + 1.0211i

ans =

  -0.6912 - 1.0211i

********************************************

// solving equation (3.23) for A=10 and B=5;

function adot=matt_dom_1(t,a)

adot=(5/a-10/a^4-1)^(1/2);

****************************************

[t,a] = ode45('matt_dom_1', [0 7.6], 1.4066)

plot(t,a)

****************************************

// plotting the matter density with time;

[t,a]=ode45('matt_dom_1',[0 7.62],1.4066)

for i=1:45

rho(i,1)=1/a(i,1)^3

end

plot(t,rho)
```

```
// plotting beta with time;

solve('-B-1+sqrt(1-B^2)=rho','B')

for i=1:45

B(i,1)=-1/2-1/2*rho(i,1)-1/2*(-rho(i,1)^2-2*rho(i,1)+1)^(1/2)

end

plot(t,B)
```

$k = 0$

```
// finding the roots of equation (3.23) for A=9 and B=4;

>>solve('4*a^3-9=0','a')

ans =

   1.3104

ans =

  -0.6552 + 1.1348i

ans =

  -0.6552 - 1.1348i

// finding the roots of equation (3.23) for A=10 and B=5;

>>solve('5*a^3-10=0','a')

ans =

   1.2599

ans =

  -0.6300 + 1.0911i
```

```
ans =

  -0.6300 - 1.0911i

// finding the roots of equation (3.23) for A=4 and B=6;

>>solve('6*a^3-4=0','a')

ans =

    0.8736

ans =

  -0.4368 + 0.7565i

ans =

  -0.4368 - 0.7565i

// solving equation (3.23) for A=10 and B=5;

****************************************

function adot=matt_dom_0(t,a)

adot=(5/a-10/a^4)^(1/2);

****************************************

>>[t,a]=ode45('matt_dom_0',[0 2],1.26)

>>plot(t,a)

****************************************

>>[t,a]=ode45('matt_dom_0',[0 2e10],1.26)

>>for i=50:145

a3(i,1)=a(i,1)^3
```

```
t2(i,1)=t(i,1)^2
end
>>plot(t2,a3)
```

****

k = -1

```
// finding the roots of equation (3.23) for A=9 and B=-4;
>> solve('-4*a^3-9+a^4=0','a')
ans =
   4.1279
ans =
  -1.2006
ans =
  0.5363 + 1.2363i
ans =
  0.5363 - 1.2363i
// finding the roots of equation (3.23) for A=10 and B=5;
>> solve('5*a^3-10+a^4=0','a')
ans =
 -0.5490 + 1.1731i
ans =
 -0.5490 - 1.1731i
```

```
ans =

   1.1744

ans =

  -5.0764
```

// finding the roots of equation (3.23) for A=4 and B=-6;

```
>> solve('-6*a^3-4+a^4=0','a')

ans =

   6.0183

ans =

  -0.8364

ans =

  0.4090 + 0.7920i

ans =

  0.4090 - 0.7920i
```

// solving equation (3.23) for A=10 and B=5; and for A=9 and B= -4

```
****************************************

function adot=matt_dom_minus1(t,a)

adot=(5/a-10/a^4+1)^(1/2)

****************************************

function adot=matt_dom_minus11(t,a)

adot=(-4/a-9/a^4+1)^(1/2)

****************************************
```

```
[t1,a1]=ode45('matt_dom_minus11',[0 10],4.128)

[t,a]=ode45('matt_dom_minus1',[0 10],1.1744)

hold on

plot(t,a)

plot(t1,a1)

hold off

// plotting the three cases;

[t,a]=ode45('matt_dom_1',[0 7.62],1.4066)

[t1,a1]=ode45('matt_dom_minus1',[0 7.62],1.1744)

[t2,a2]=ode45('matt_dom_0',[0 7.62],1.26)

hold on

plot(t,a)

plot(t1,a1)

plot(t2,a2)

hold off

*****************************************
```

3. Spherically symmetric solution

```
// plotting solution (2.30)

>> [X,Y] = meshgrid(-38.2525:4.5003:38.2525);

>> for i=1:17

for j=1:17
```

```
s(i,j)=X(i,j)^2+Y(i,j)^2;
r(i,j)=sqrt(s(i,j));
a(i,j)=r(i,j)-3;
b(i,j)=sqrt(a(i,j));
c(i,j)=3.4641*b(i,j);
d(i,j)=-3.4641*b(i,j);
end
end
>> surf(X,Y,c)
>> surf(X,Y,d)
```

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